Solving ~~QFTs~~ convection–diffusion equations with finite volume methods

Kurganov and Tadmor (KT) $O(x^2)$ central scheme - An appetizer

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QCD & beyond with the functional renormalisation group,
Heidelberg, July 17, 2019

- FRG flow eqs. especially LPA flow eqs. of low-energy effective models can be written as **Convection–diffusion equations**[1]:

$$\partial_t u(t,x) + \partial_x F[t, u(t,x)] = \partial_x Q[t, x, u(t,x), \partial_x u(t,x)] + \partial_x S(t,x)$$

  - Very common class of PDEs in Physics, Engineering and Numerical Mathematics
  - Well established numerical methods available (see talks of N. Wink and D. Rischke) $\Rightarrow$ among them are **Finite volume (FV) methods**

[1] E. Grossi and N. Wink (2019), arXiv: 1903.09503 [hep-th]

Convection–diffusion equations

$$\partial_t u(t,x) + \partial_x F[t, u(t,x)] = \partial_x Q[t, x, u(t,x), \partial_x u(t,x)] + \partial_x S(t,x) \quad (1)$$

- $u(t,x)$ vector of conserved quantities
- $F[t, u(t,x)]$ nonlinear convection flux
- $Q[t, x, u(t,x), \partial_x u(t,x)]$ dissipation flux
- $S(t,x)$ source term

Examples:

- Linear Advection and (inviscid/viscous) Burgers' eqs. with non-smooth ICs
- Euler Equations of Gas Dynamics (Shock tube problem)
- **FRG LPA flow equations**

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New High-Resolution Central Schemes for Nonlinear Conservation Laws and Convection–Diffusion Equations

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A. Kurganov and E. Tadmor, J. Comput. Phys. **160** (May 2000)

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Semi-discrete central scheme

- Equidistant spatial finite volume grid with n cells

$$\{x_j\} = \{x_0, x_0 + \Delta x, \ldots, x_1\} \tag{2}$$
$$\{u_j\} = \{u(t, x_0), u(t, x_0 + \Delta x), \ldots, u(t, x_1)\} \tag{3}$$

with extrapolated ghost points e.g. $u_{-1} = 2u_0 - u_1$

- Conservation form:

$$\frac{\mathrm{d}u_j}{\mathrm{d}t} = -\frac{H_{j+1/2}(t) - H_{j-1/2}(t)}{\Delta x} + \frac{P_{j+1/2}(t) - P_{j-1/2}(t)}{\Delta x} + \frac{S_{j+1/2}(t) - S_{j-1/2}(t)}{\Delta x} \tag{4}$$

  - $H_{j\pm1/2}(t)$ Numerical convection flux on left/right cell boundary
  - $P_{j\pm1/2}(t)$ Numerical diffusion flux on left/right cell boundary
  - $S_{j\pm1/2}(t)$ Numerical source term on left/right cell boundary

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KT $O(x^2)$ - Numerical convection flux $H_{j+1/2}(t)$

$$H_{j+1/2}(t) = \frac{F[t, u^+_{j+1/2}] + F[t, u^-_{j+1/2}]}{2} - \frac{a_{j+1/2}}{2}\left(u^+_{j+1/2} - u^-_{j+1/2}\right), \tag{5}$$

- Maximal local speed

$$a_{j+1/2} = \max\left(\rho\left(\frac{\partial F}{\partial u}(u^+_{j+1/2})\right), \rho\left(\frac{\partial F}{\partial u}(u^-_{j+1/2})\right)\right) \tag{6}$$

- Approximate derivatives (TVD reconstruction) with limiter ϕ

$$u^+_{j+1/2} = u_{j+1} - \frac{\Delta x}{2}\partial_x u_{j+1} = u_{j+1} - \frac{1}{2}\phi\left(\frac{u_{j+1} - u_j}{u_{j+2} - u_{j+1}}\right)(u_{j+2} - u_{j+1}) \tag{7}$$

$$u^-_{j+1/2} = u_j + \frac{\Delta x}{2}\partial_x u_j \qquad = u_j + \frac{1}{2}\phi\left(\frac{u_j - u_{j-1}}{u_{j+1} - u_j}\right)(u_{j+1} - u_j) \tag{8}$$

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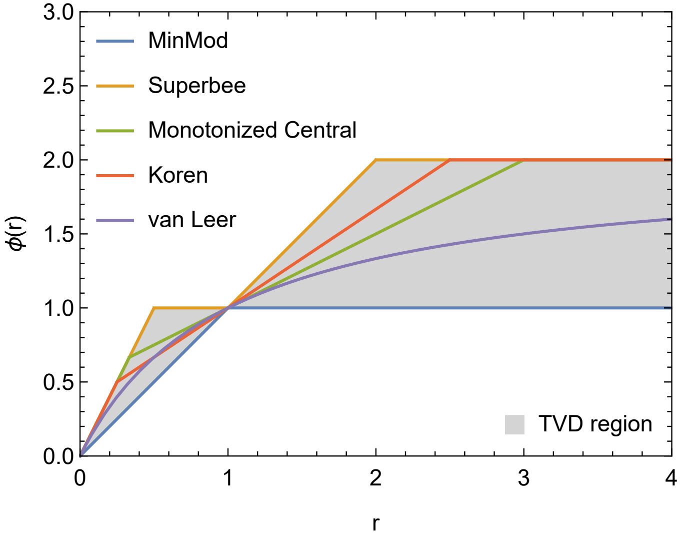

expressions from: en.wikipedia.org/wiki/Flux_limiter - 2019.07.10 - 16:30

KT $O(x^2)$ - $P_{j+1/2}(t)$ and $S_{j+1/2}(t)$

- Central difference approximation of diffusion flux:

$$P_{j+1/2}(t) = \frac{1}{2}Q[t, x_j, u_j, (u_{j+1} - u_j)/\Delta x] + \\ + \frac{1}{2}Q[t, x_{j+1}, u_{j+1}, (u_{j+1} - u_j)/\Delta x] \tag{9}$$

  - Diffusion flux for $\dot{u}_j$ is based on the 3-point stencil $\{u_{j-1}, u_j, u_{j+1}\}$

- Analogous central difference approximation of source flux:

$$S_{j+1/2}(t) = \frac{1}{2}S(t, x_j) + \frac{1}{2}S(t, x_{j+1}) \tag{10}$$

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KT $O(x^2)$ its in semi-discrete form - Summary

- Method of Lines (MoL) finite volume PDE discretization $\Rightarrow$ sparse (banded) ODE system
- Total RHS flux for $\dot{u}_j$ is based on the 5-point stencil $\{u_{j-2}, u_{j-1}, u_j, u_{j+1}, u_{j+2}\}$
- Approximate derivatives are reconstructed from the computed cell averages using ϕ-Limiter
- Only additional information (apart from PDE and grid): spectral radius of jacobian $\rho = \max|\lambda_i(\partial F/\partial u)|$ to approximate local speeds

Godunov first order upwind scheme

$$\frac{\mathrm{d}u_j}{\mathrm{d}t} = -\frac{F[t,u_{j+1}]-F[t,u_j]}{\Delta x} + \frac{Q[t,x_{j+1},u_{j+1},(u_{j+1}-u_j)/\Delta x]-Q[t,x_j,u_j,(u_j-u_{j-1})/\Delta x]}{\Delta x} + \frac{S[t,x_{j+1}]-S[t,x_j]}{\Delta x} \tag{11}$$

- Applicable only to convection–diffusion eqs. with $\partial F/\partial u < 0 \; \forall \; t, u$
  - Linear Advection eq. with negative velocity
  - FRG LPA flow eqs.
  - ...

Classical examples

- Linear Advection eq.:

$$\partial_t u(t,x) - \partial_x u(t,x) = 0, \qquad u(0,x) = \begin{cases} 1 & 3 \le x \le 4 \\ 0 & \text{otherwise} \end{cases} \tag{LA}$$

- Burgers' eq.:

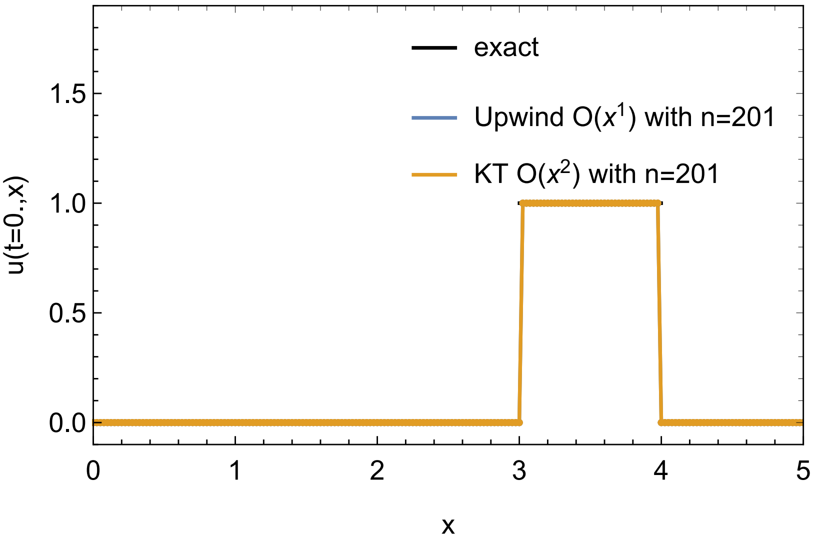
$$\partial_t u(t,x) + \partial_x \frac{u(t,x)^2}{2} = \nu \partial_x^2 u(t,x) \tag{B}$$

$$u(0,x) = 0.5 + \sin(x) \quad \text{and inviscid:} \quad \nu = 0 \tag{B.1}$$

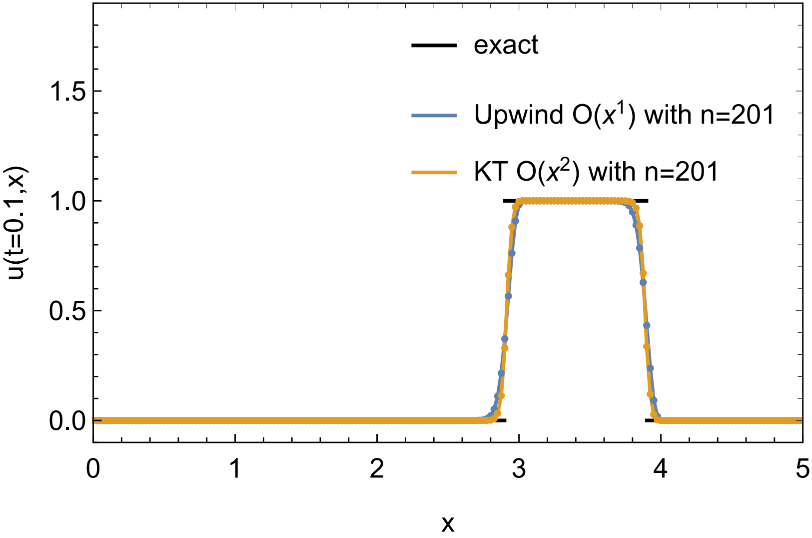
$$u(0,x) = 0.5 + \sin(x) \quad \text{and viscous:} \quad \nu = 0.1 \tag{B.2}$$

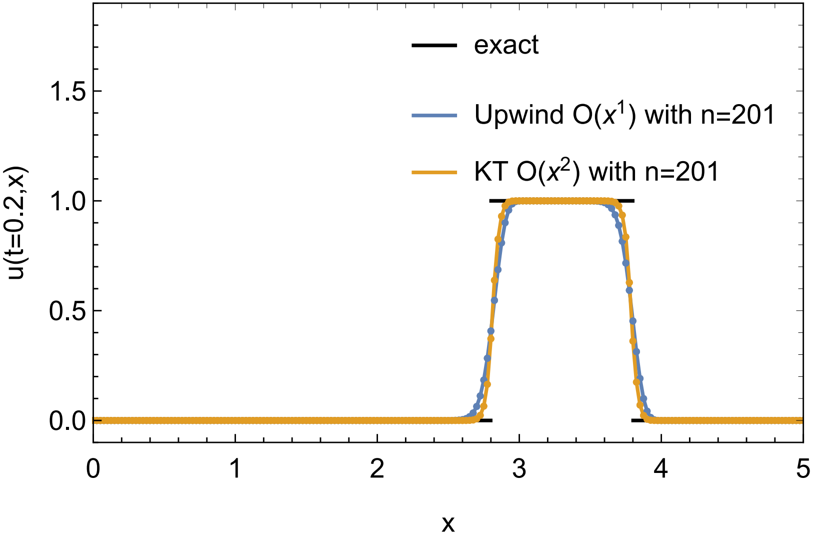
$$u(0,x) = \begin{cases} 1 & x \le 2 \\ 0 & \text{otherwise} \end{cases} \quad \text{and viscous:} \quad \nu = 0.01 \tag{B.3}$$

Linear Advection eq. (LA)

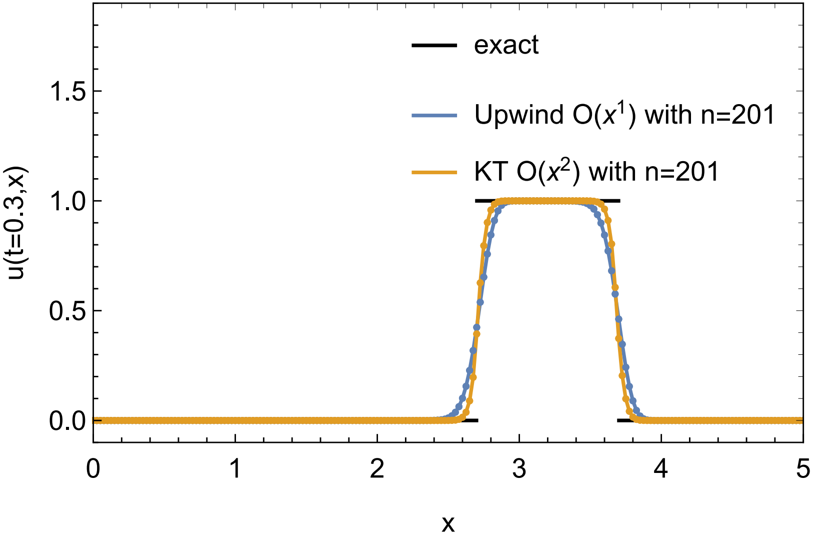


Linear Advection eq. (LA)

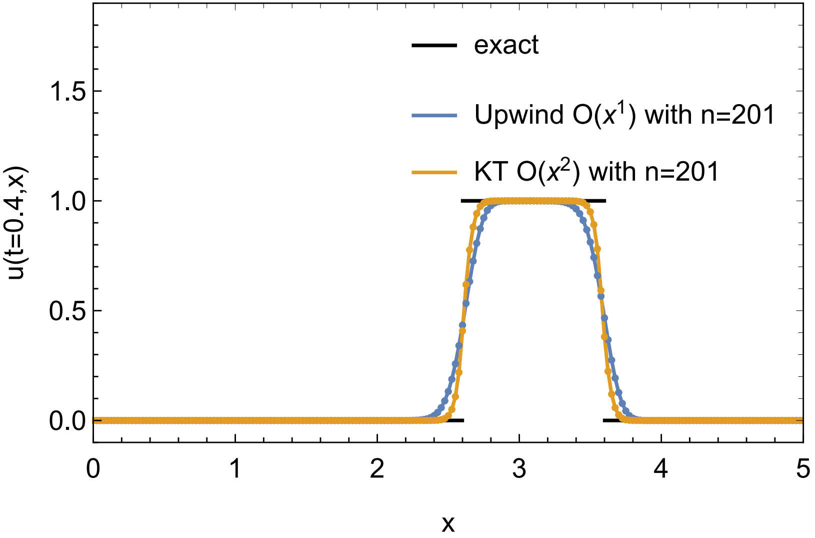



exact
Upwind O(x^1) with n=201
KT O(x^2) with n=201
u(t=0.2,x)
x
0.0
0.5
1.0
1.5
0
1
2
3
4
5

Linear Advection eq. (LA)



Linear Advection eq. (LA)



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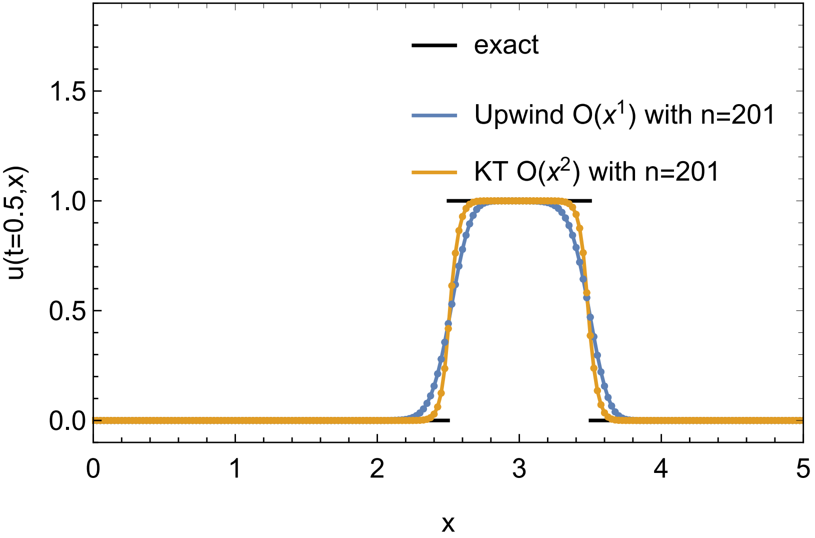
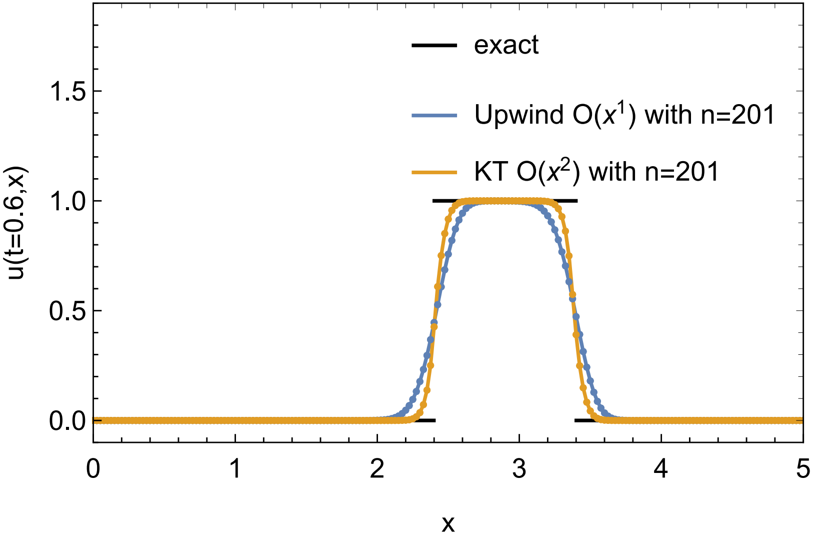
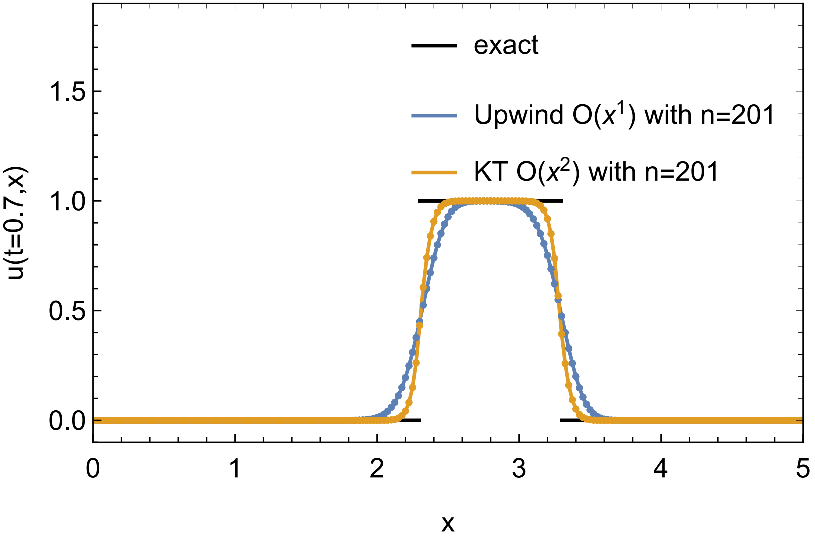
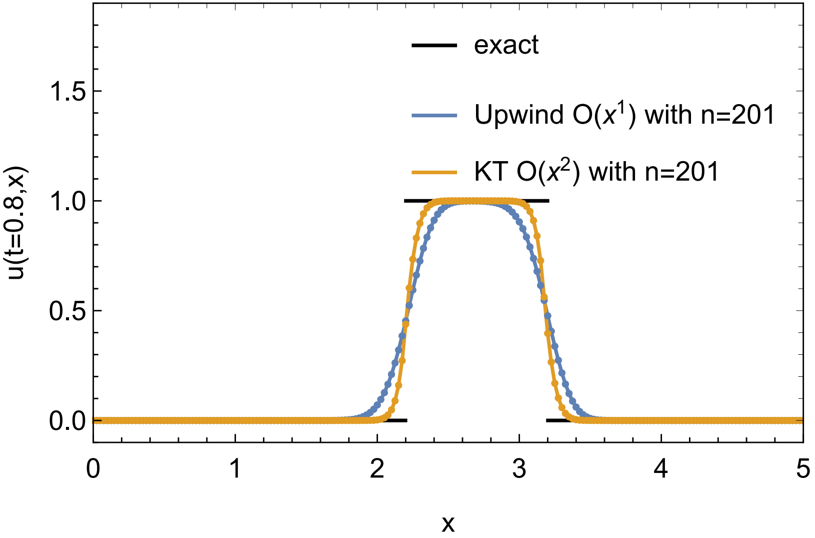


Linear Advection eq. (LA)

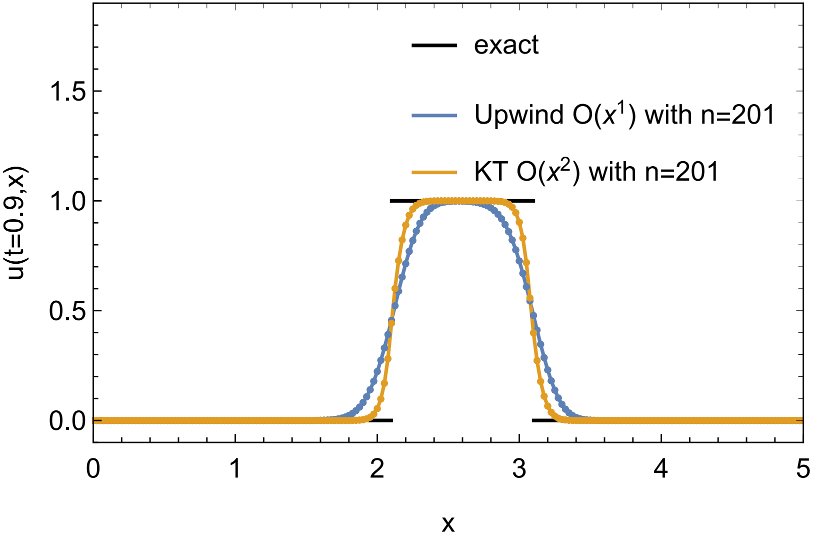


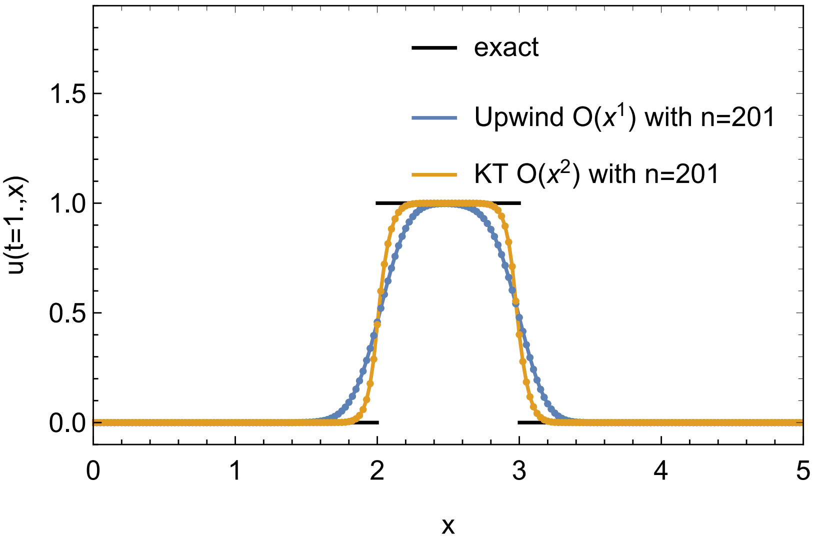

exact
Upwind O(x^1) with n=201
KT O(x^2) with n=201
u(t=0.7,x)
x
0.0
0.5
1.0
1.5
0
1
2
3
4
5

Linear Advection eq. (LA)

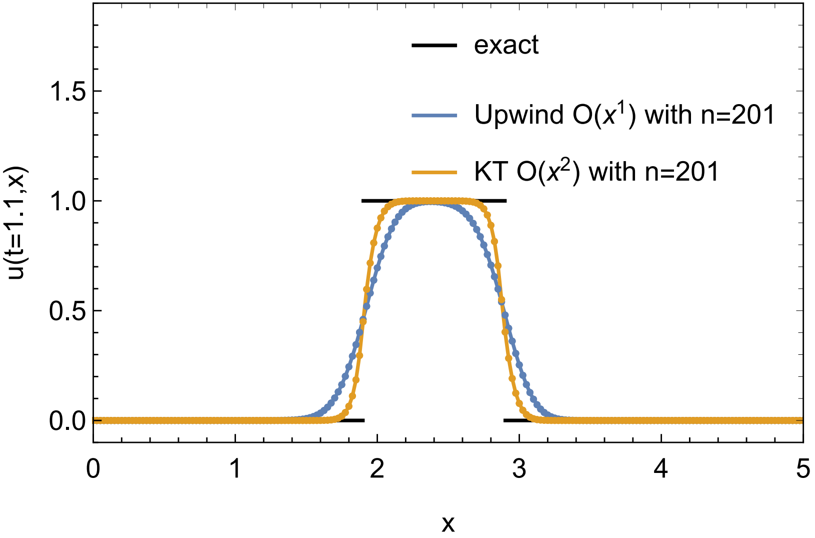


Linear Advection eq. (LA)




exact
Upwind O(x^1) with n=201
KT O(x^2) with n=201
1.5
1.0
0.5
0.0
u(t=1.,x)
0
1
2
3
4
5
x

Linear Advection eq. (LA)



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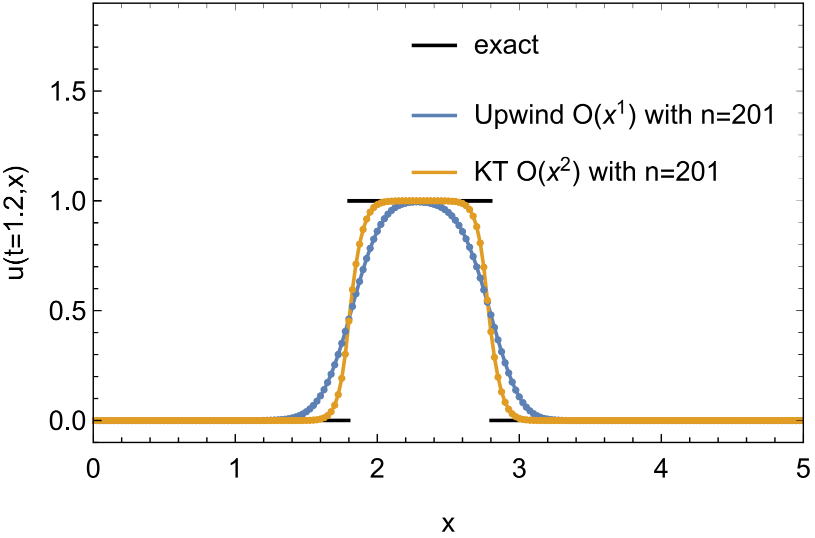

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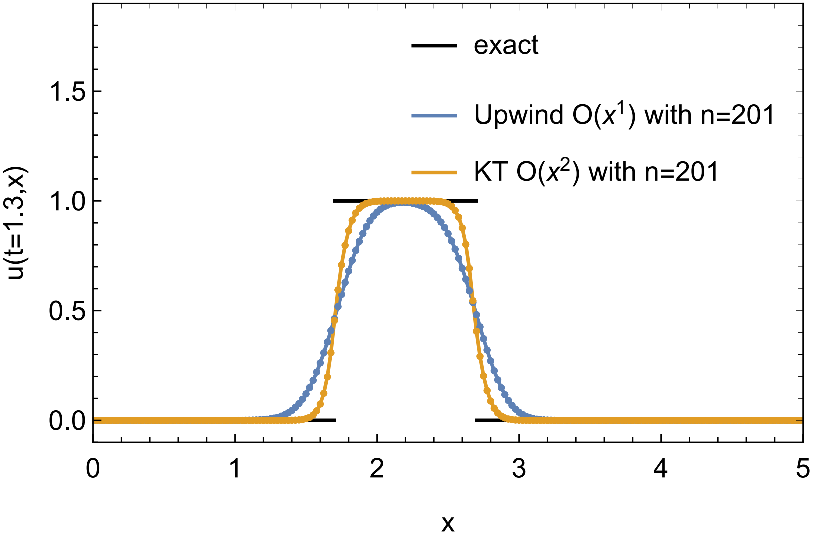

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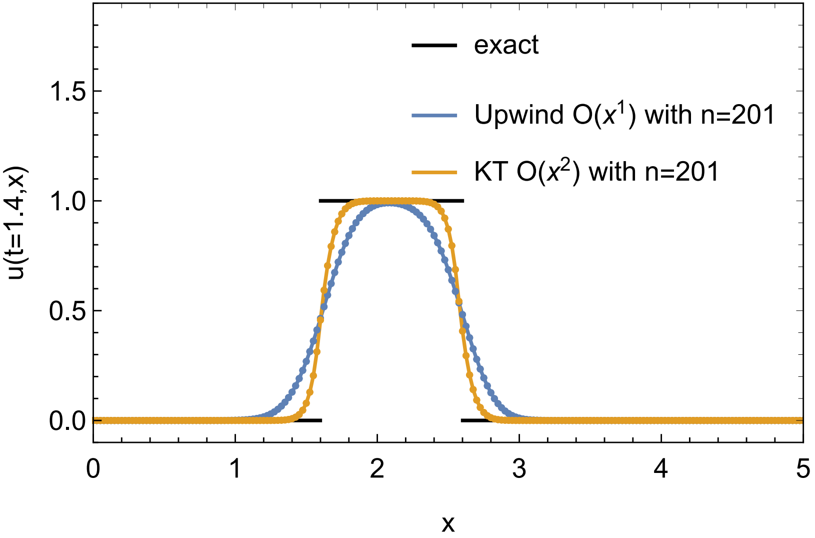
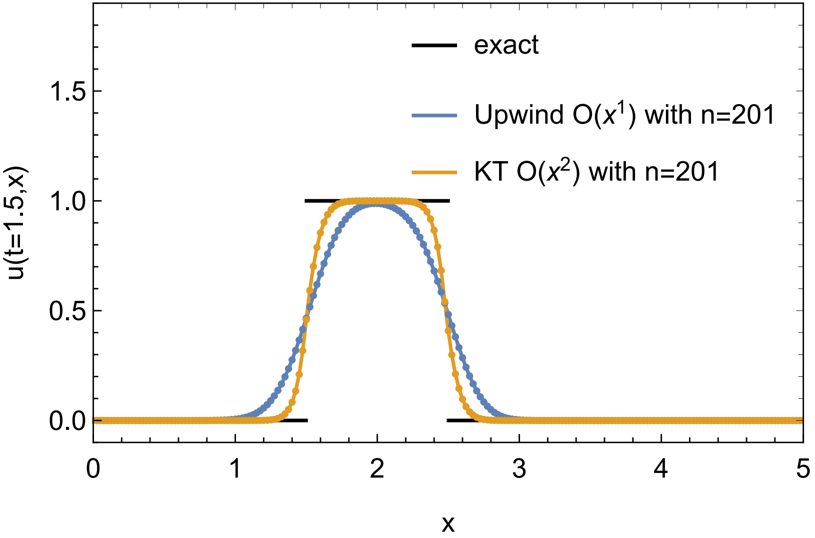


Linear Advection eq. (LA)



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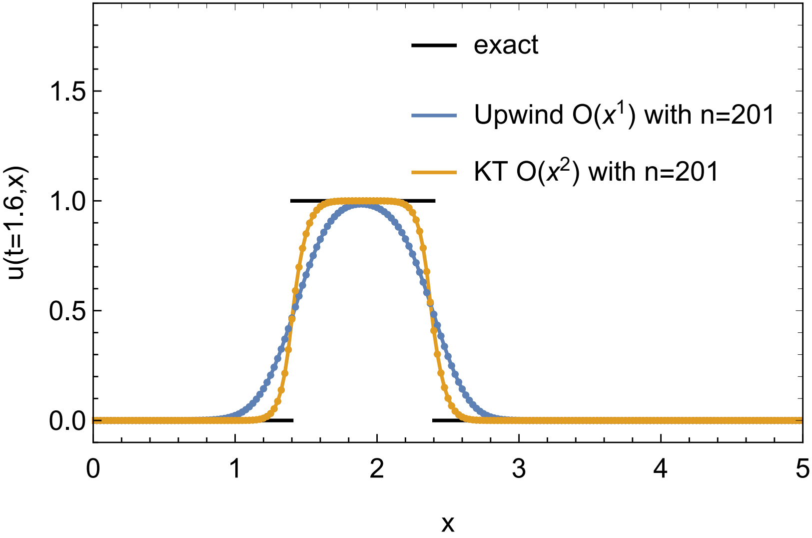

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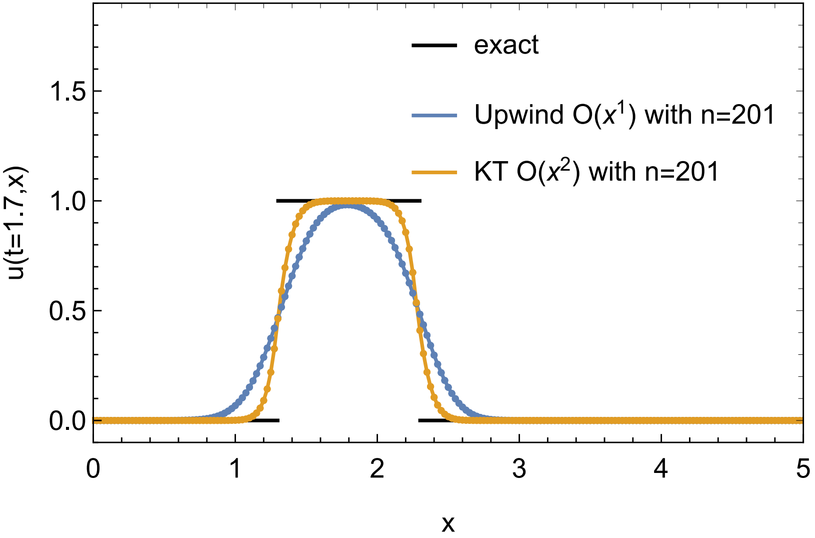

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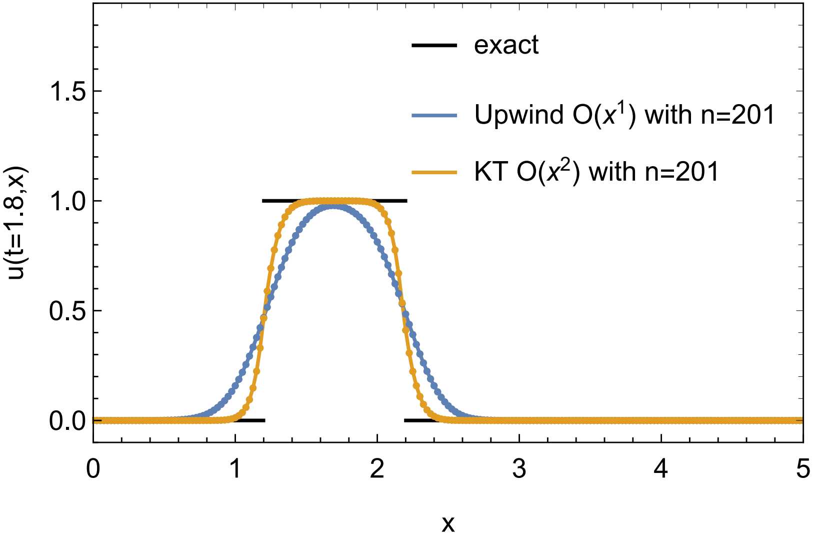
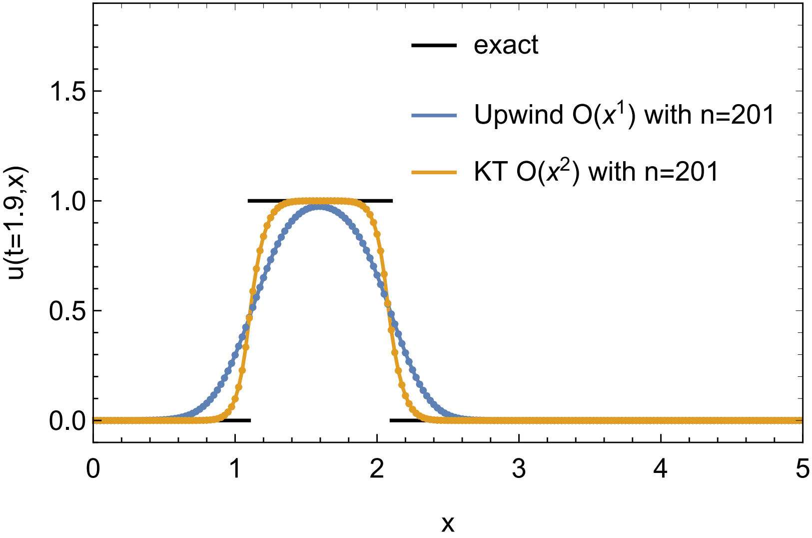
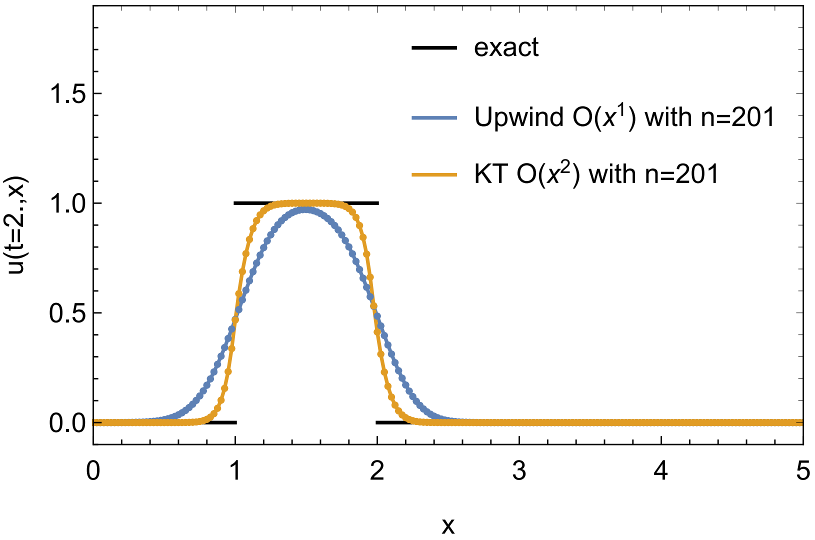


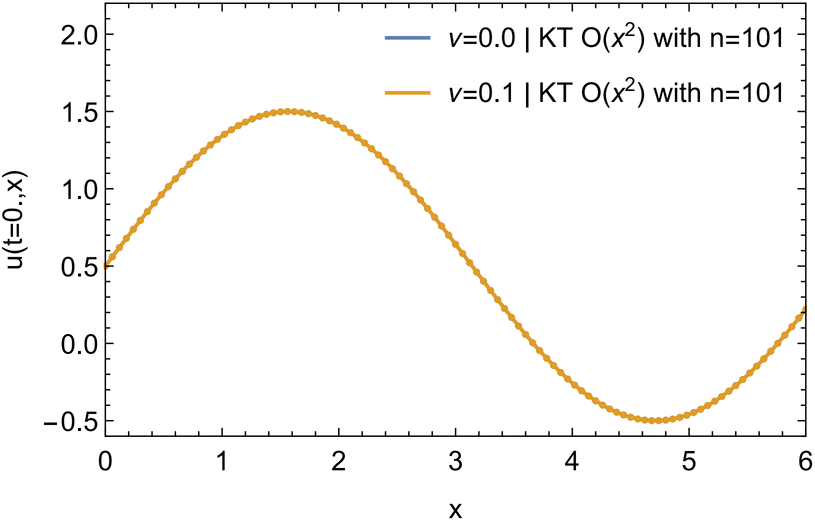
Linear Advection eq. (LA)



Linear Advection eq. (LA)



Burgers' eq. (B1) and (B2)



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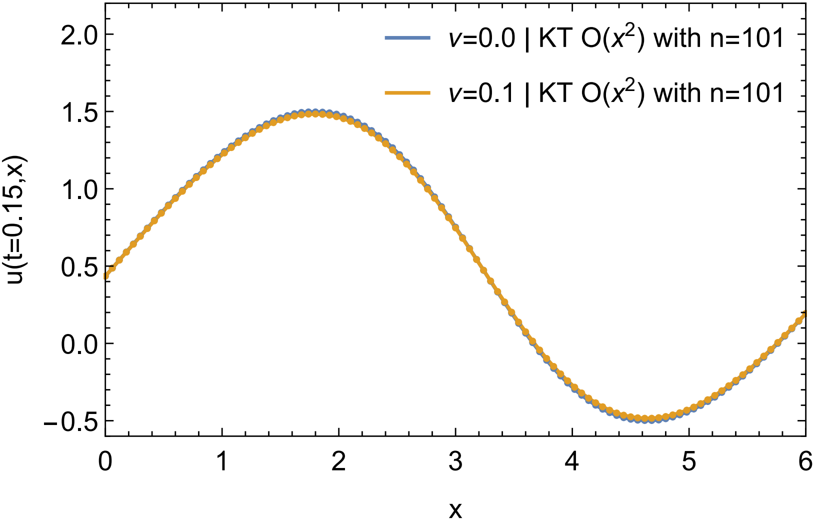

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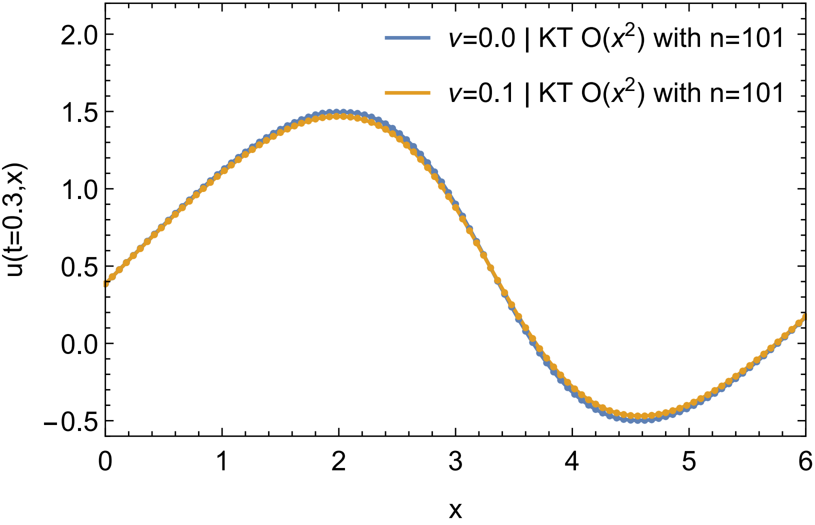

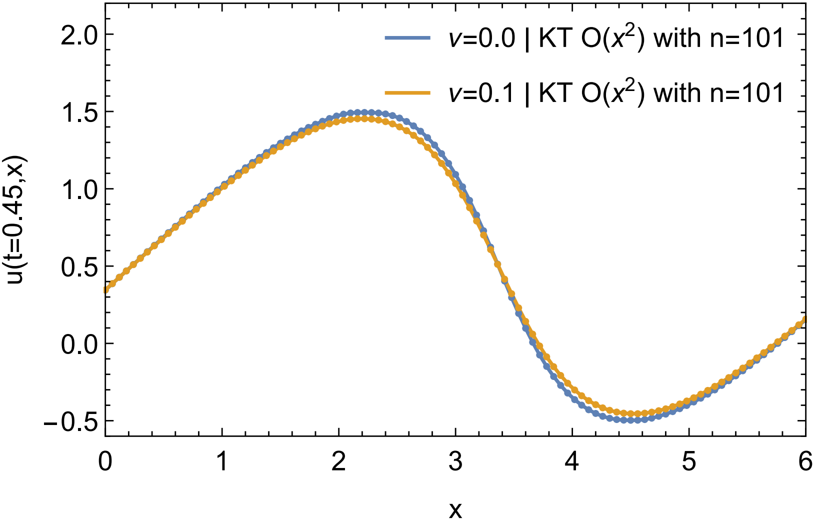
v=0.0 | KT O(x^2) with n=101
v=0.1 | KT O(x^2) with n=101
u(t=0.45,x)
x
2.0
1.5
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0.5
0.0
−0.5
0
1
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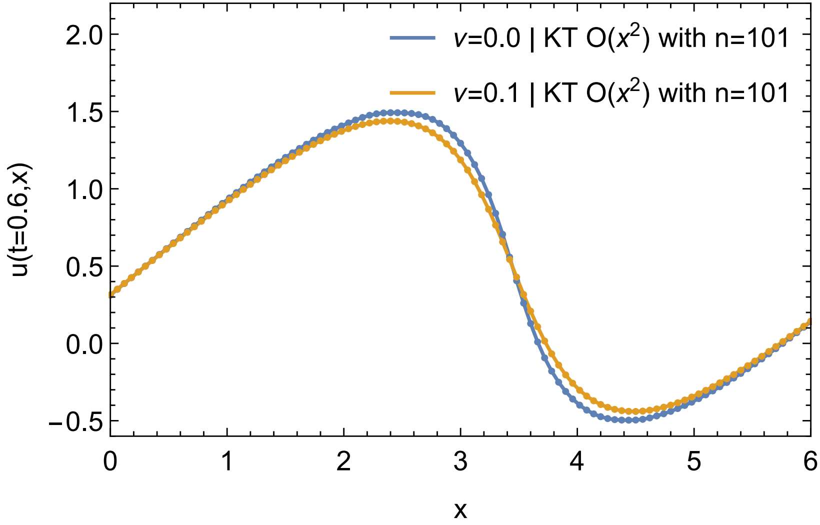
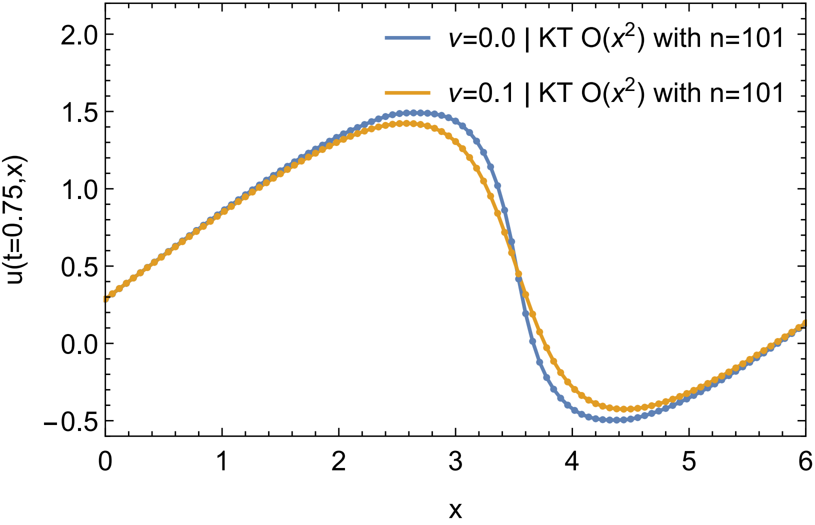
6

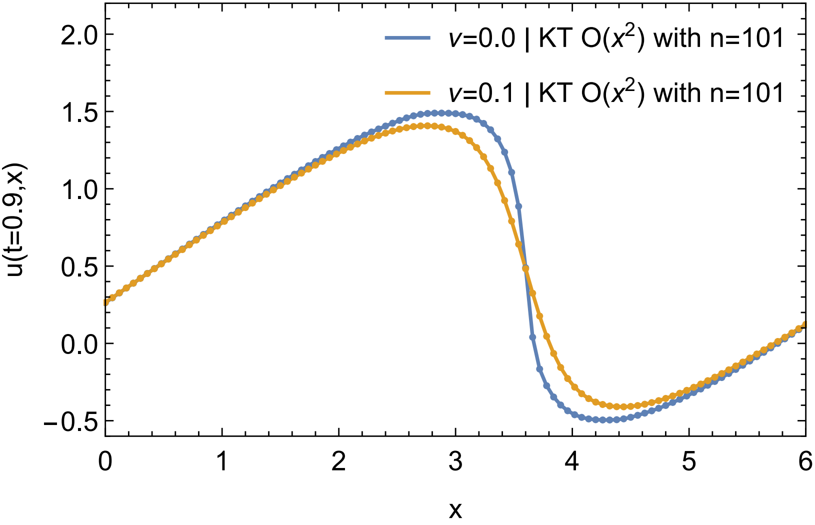
Burgers' eq. (B1) and (B2)



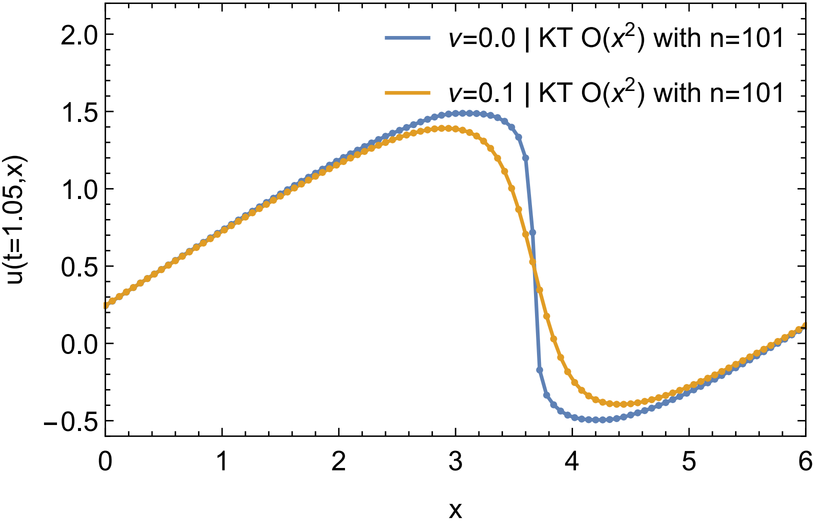
Burgers' eq. (B1) and (B2)



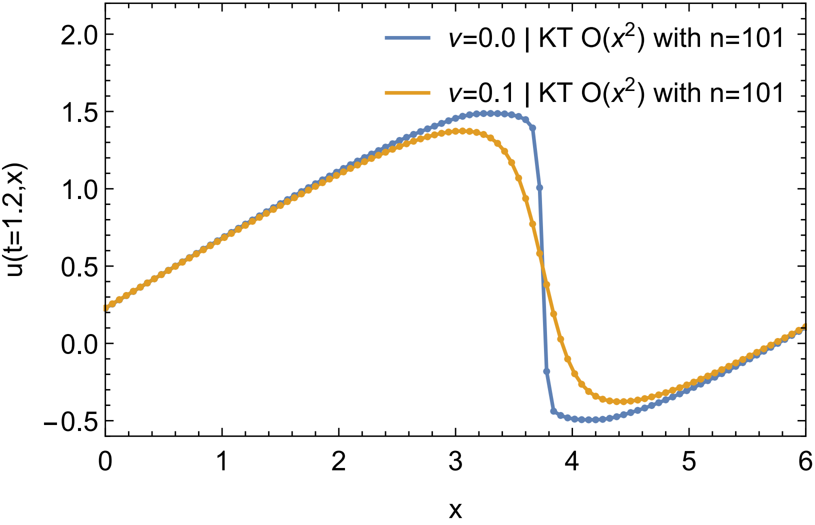
Burgers' eq. (B1) and (B2)



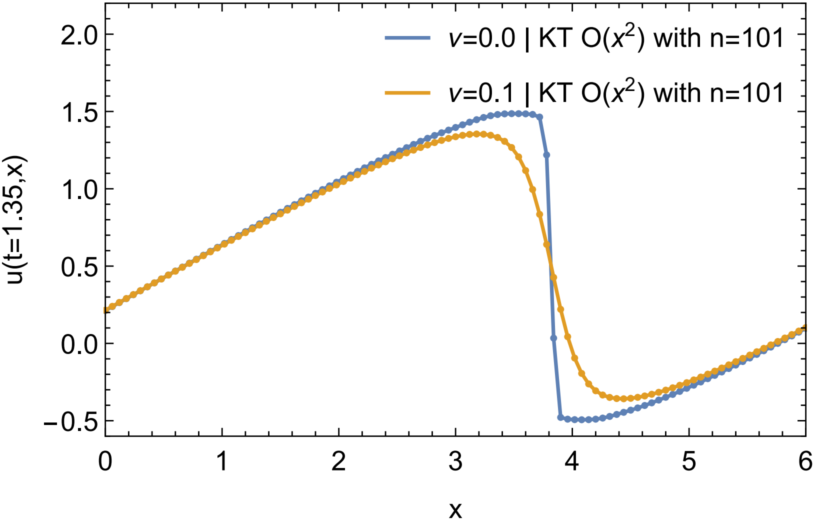
Burgers' eq. (B1) and (B2)

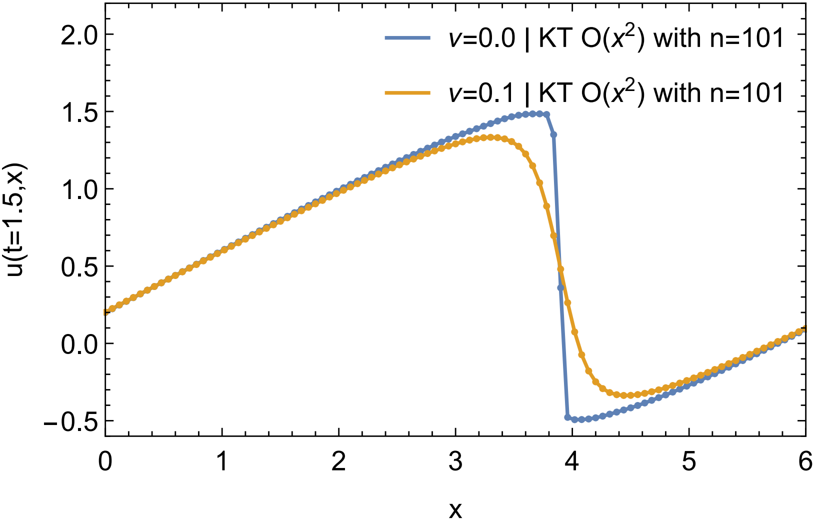


Burgers' eq. (B1) and (B2)



Burgers' eq. (B1) and (B2)





v=0.0 | KT O(x^2) with n=101
v=0.1 | KT O(x^2) with n=101
u(t=1.5,x)
x
2.0
1.5
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0.0
−0.5
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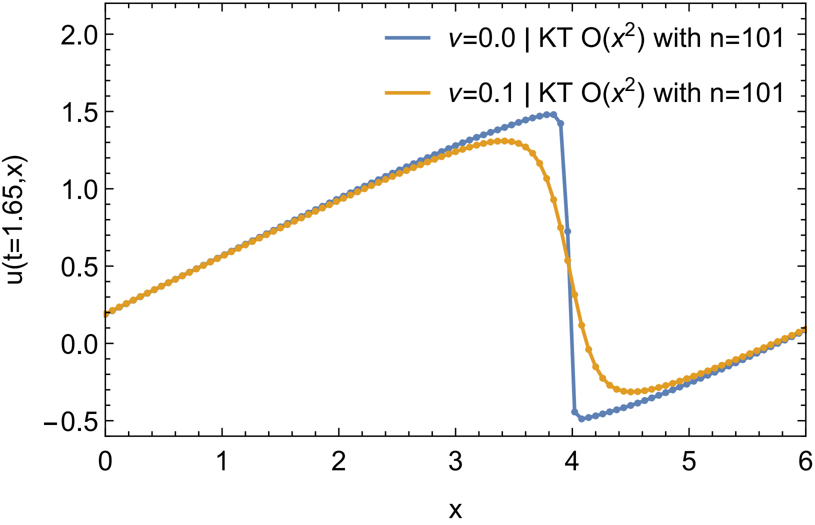
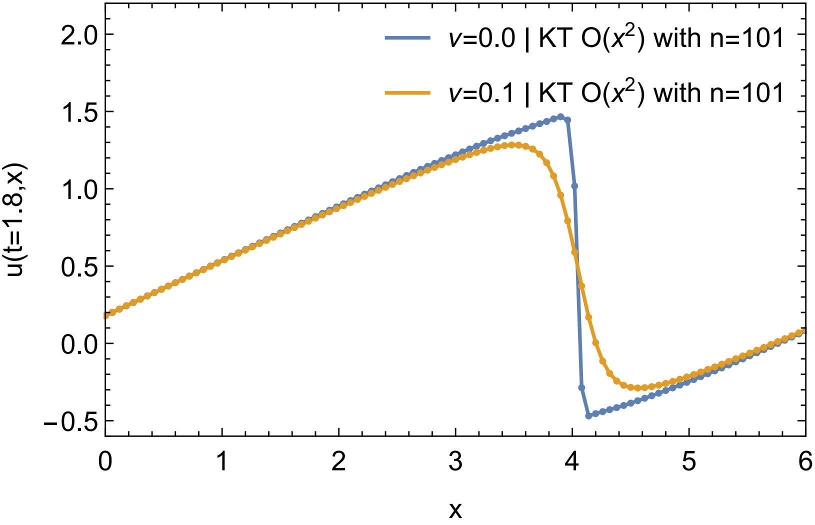
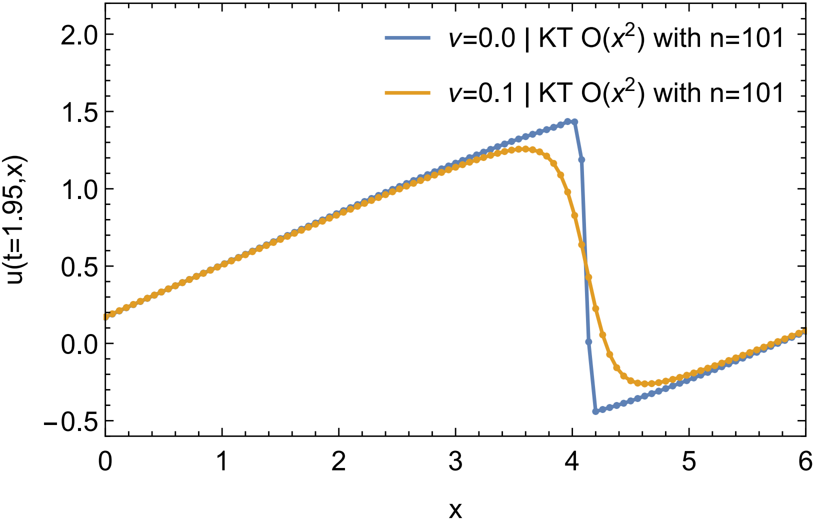
Burgers' eq. (B1) and (B2)



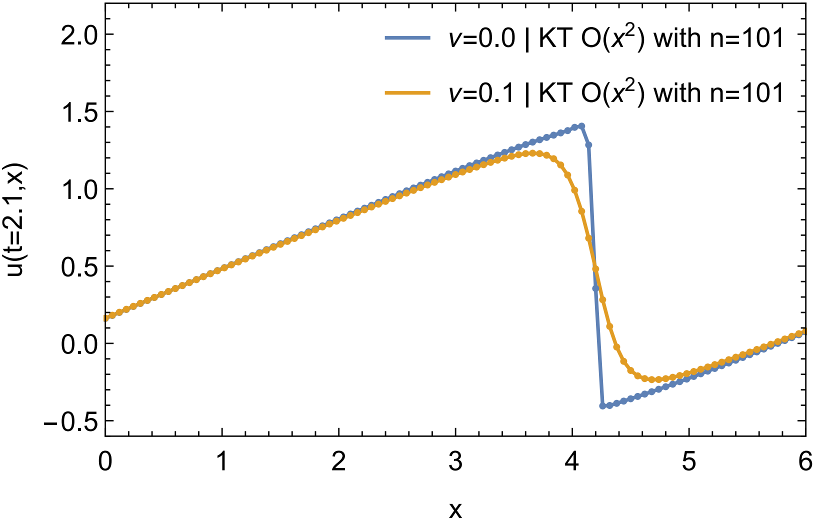
Burgers' eq. (B1) and (B2)



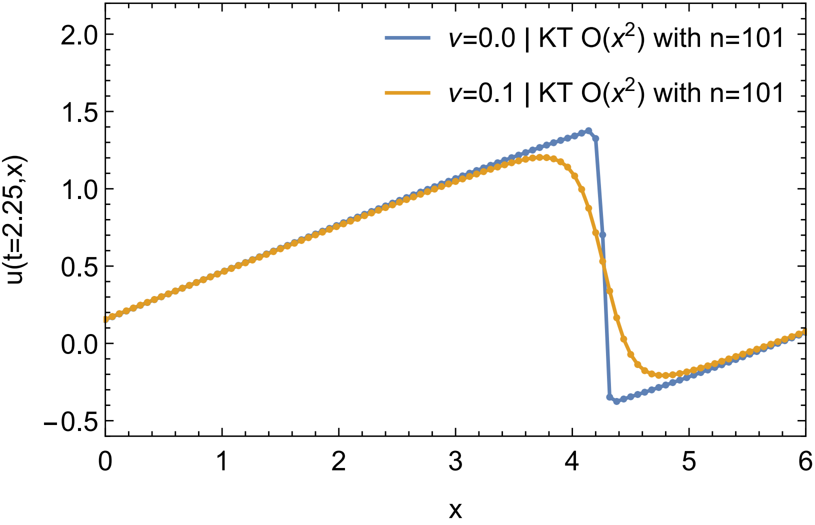
Burgers' eq. (B1) and (B2)



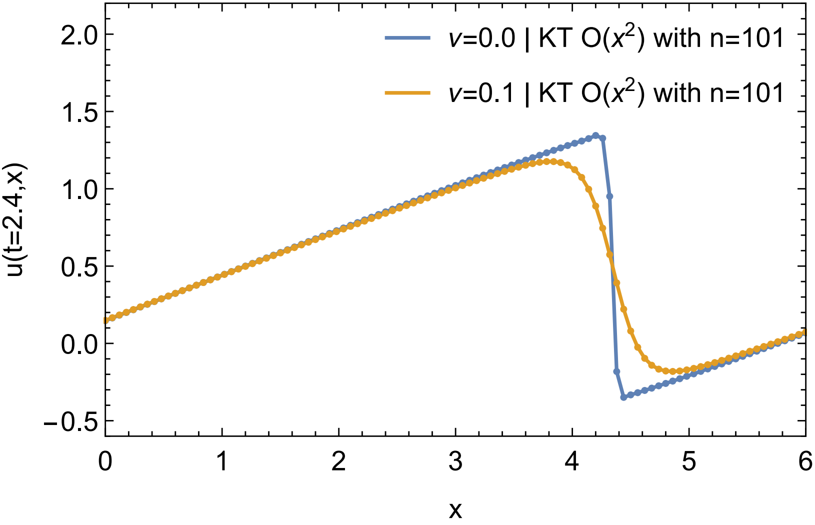
Burgers' eq. (B1) and (B2)



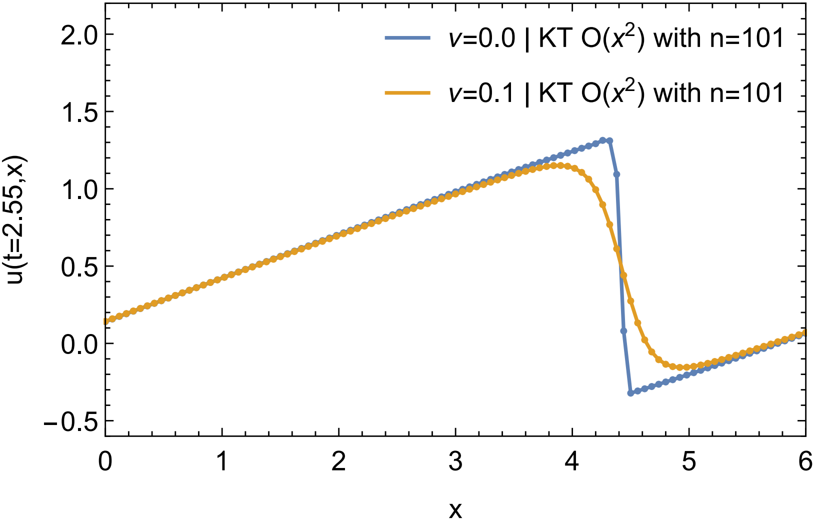
Burgers' eq. (B1) and (B2)



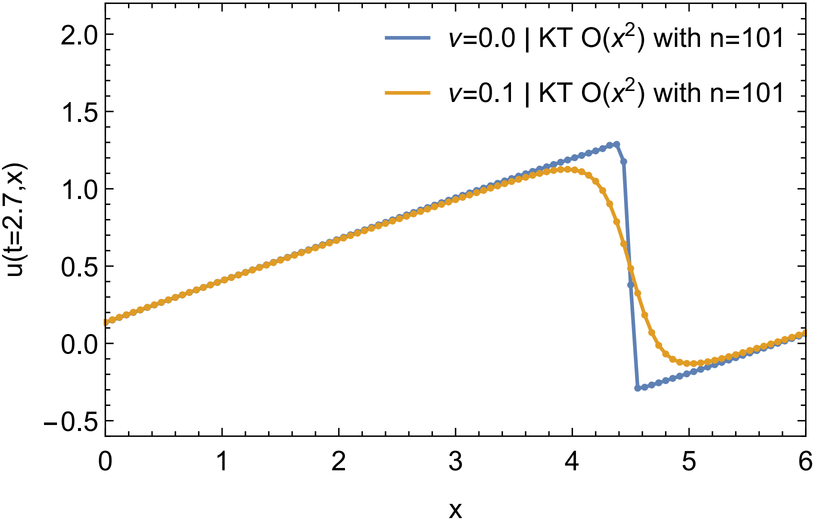
Burgers' eq. (B1) and (B2)



Burgers' eq. (B1) and (B2)



Burgers' eq. (B1) and (B2)



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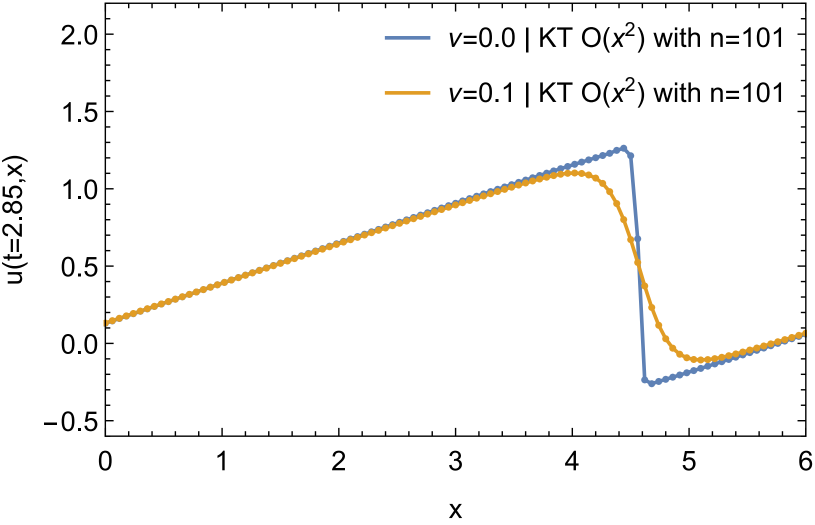
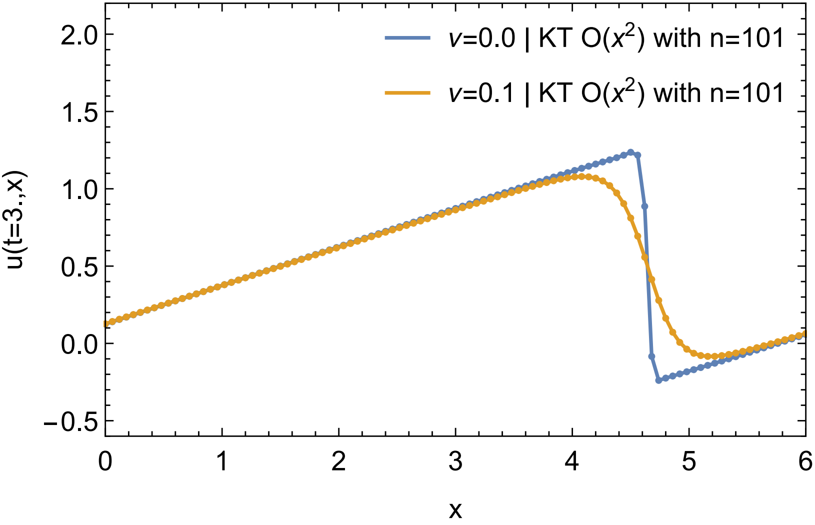
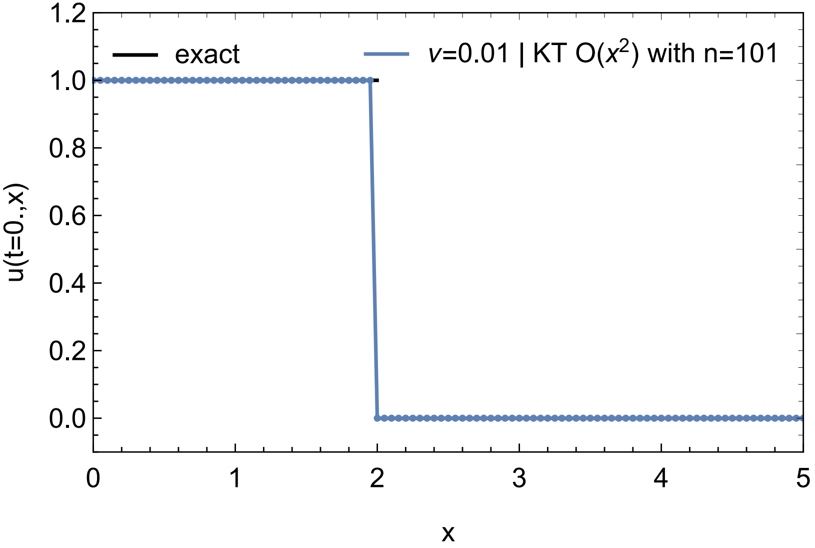
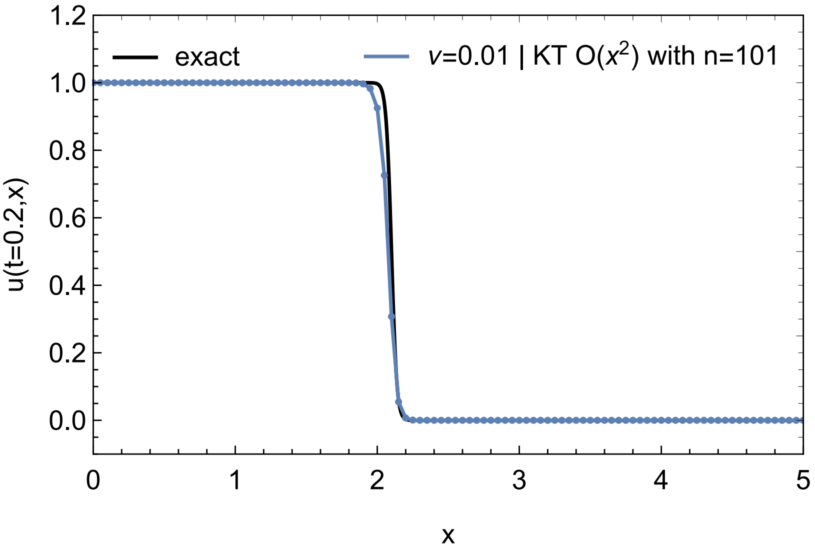


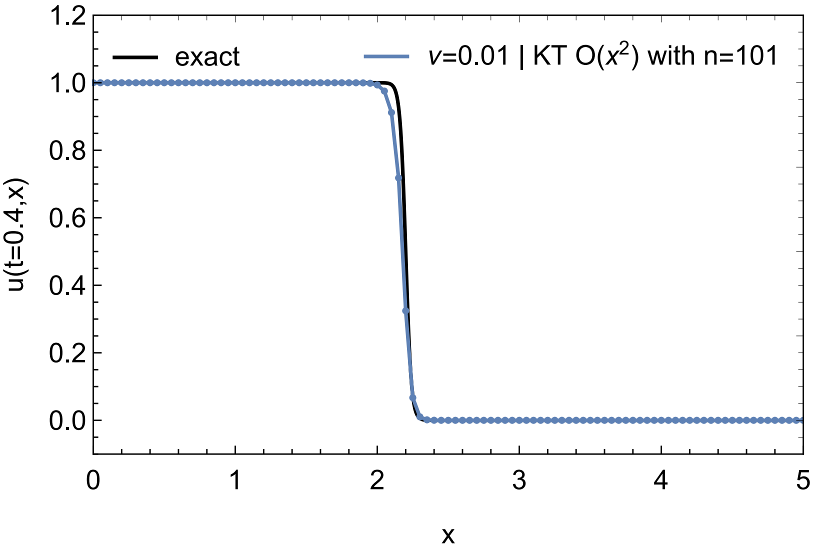
Burgers' eq. (B1) and (B2)



Burgers' eq. (B3)

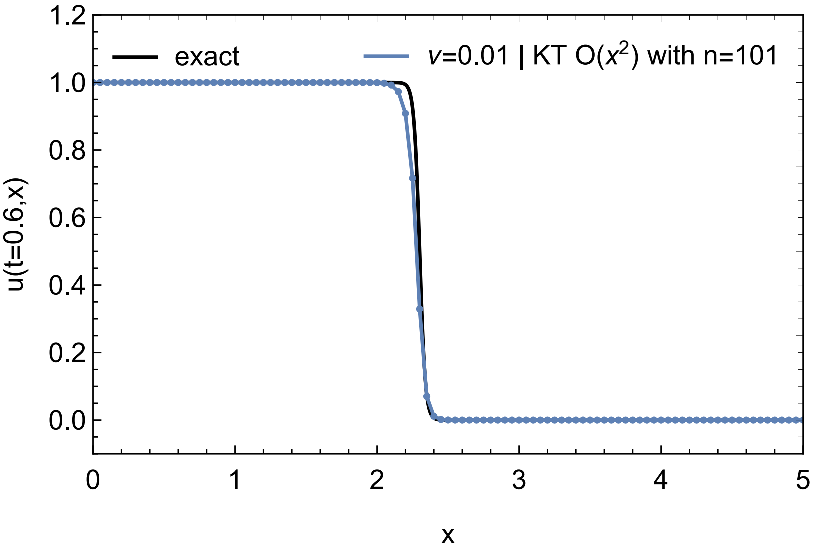



exact
v=0.01 | KT O(x²) with n=101
u(t=0.2,x)
x
1.2
1.0
0.8
0.6
0.4
0.2
0.0
0
1
2
3
4
5

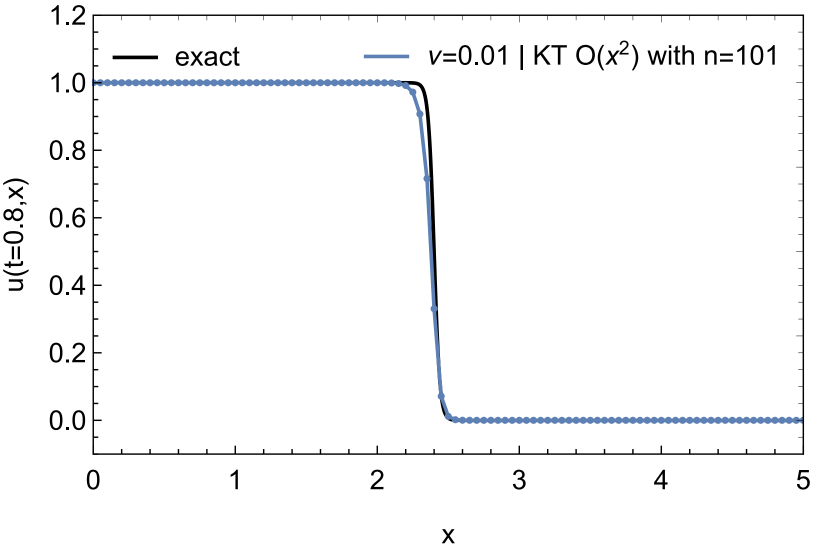


exact
v=0.01 | KT O(x2) with n=101
u(t=0.4,x)
x
1.2
1.0
0.8
0.6
0.4
0.2
0.0
0
1
2
3
4
5

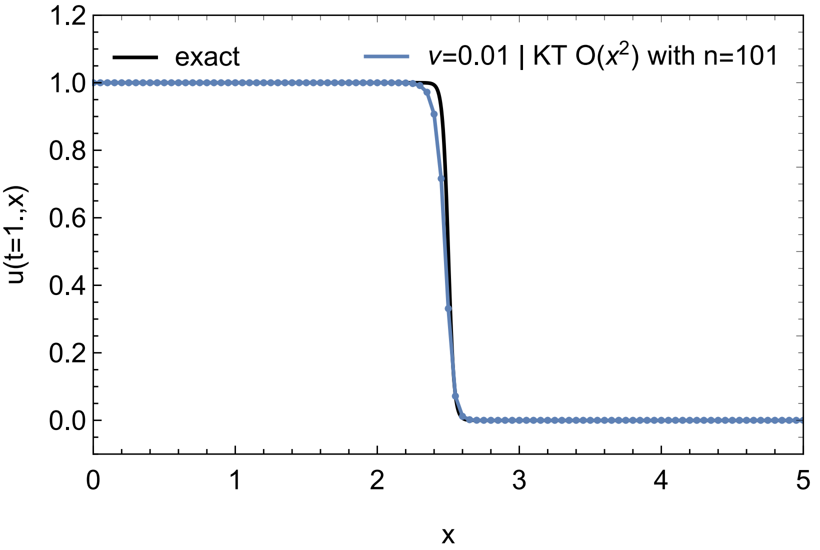
Burgers' eq. (B3)

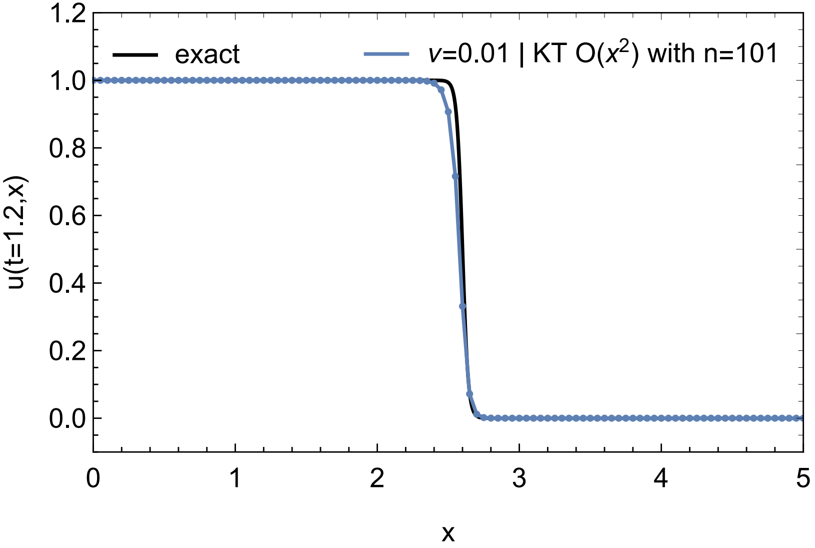


Burgers' eq. (B3)

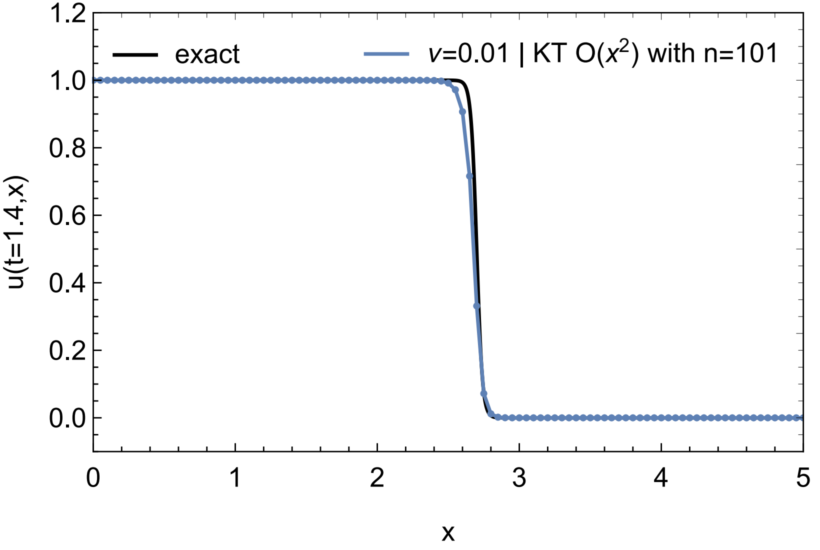


Burgers' eq. (B3)

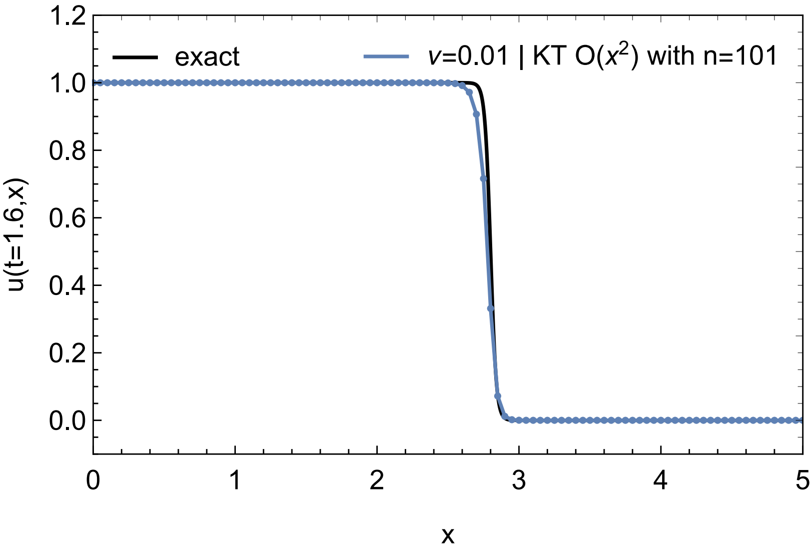



exact
v=0.01 | KT O(x2) with n=101
u(t=1.2,x)
x
1.2
1.0
0.8
0.6
0.4
0.2
0.0
0
1
2
3
4
5

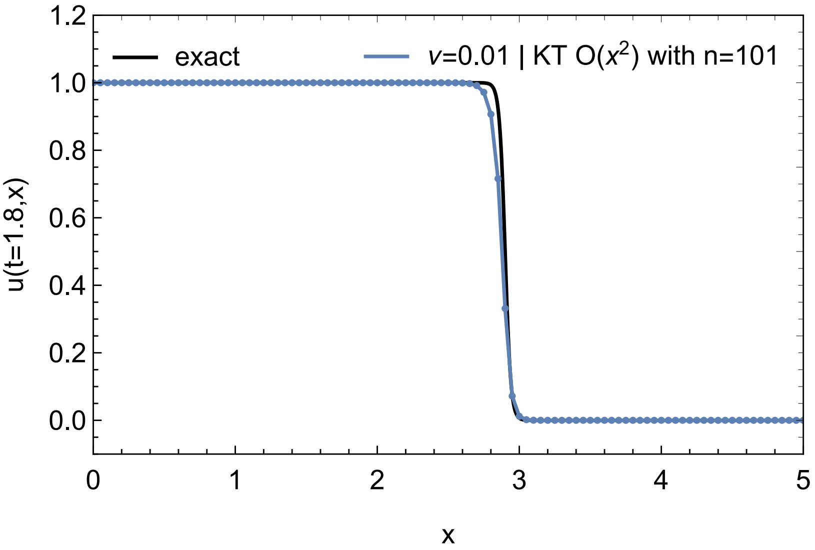
Burgers' eq. (B3)

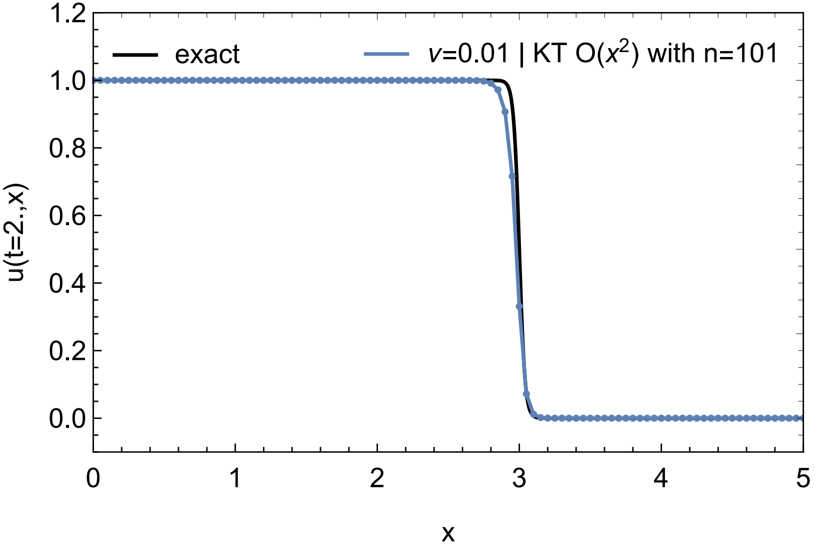


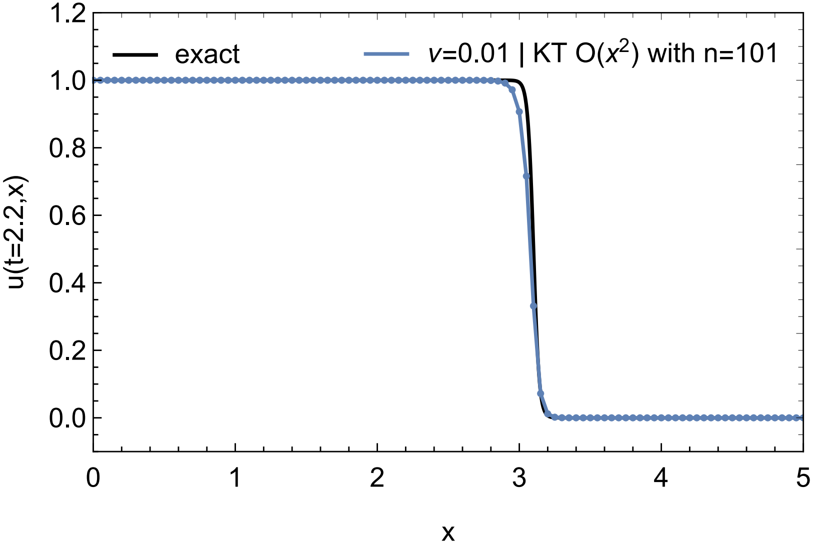
Burgers' eq. (B3)



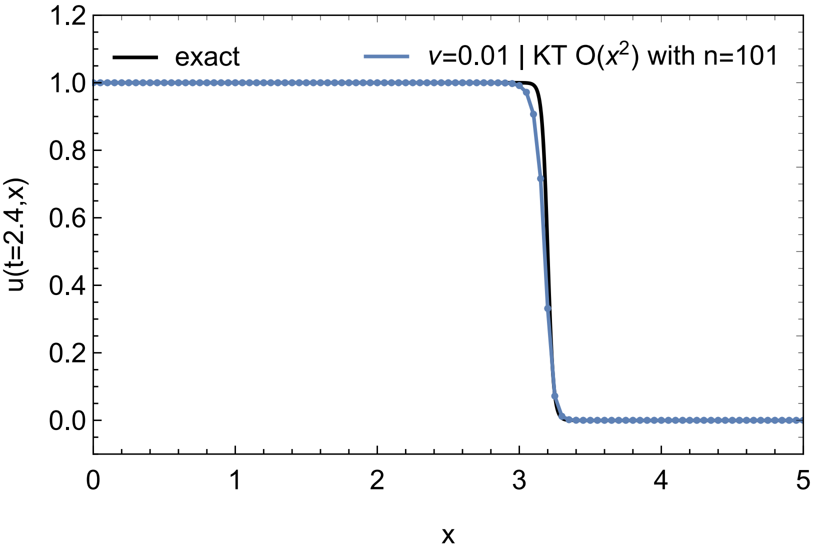
Burgers' eq. (B3)



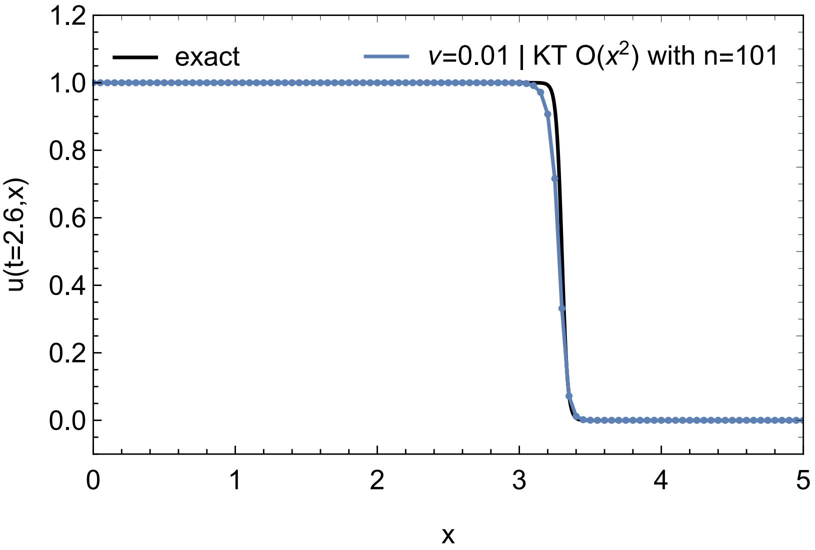

1.2
1.0
0.8
0.6
0.4
0.2
0.0
u(t=2.,x)
0
1
2
3
4
5
x
exact
ν=0.01 | KT O(x²) with n=101

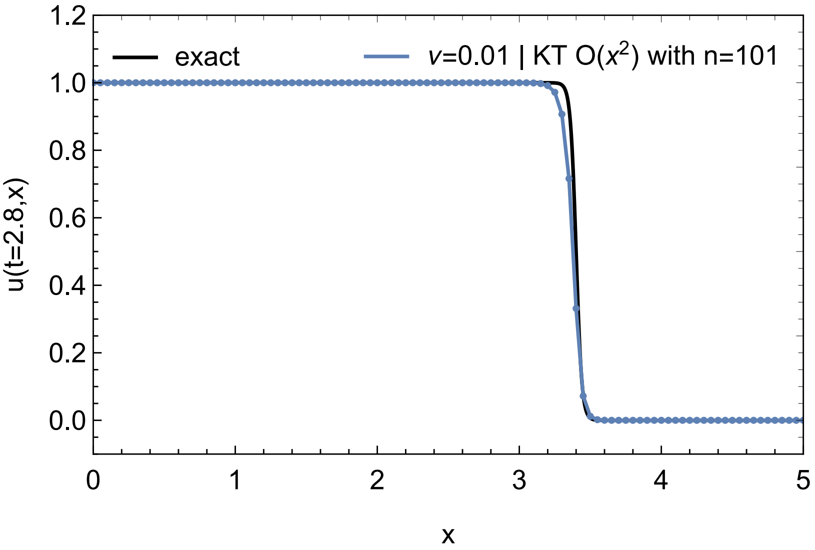

exact
v=0.01 | KT O(x2) with n=101
1.2
1.0
0.8
0.6
0.4
0.2
0.0
u(t=2.2,x)
0
1
2
3
4
5
x

Burgers' eq. (B3)



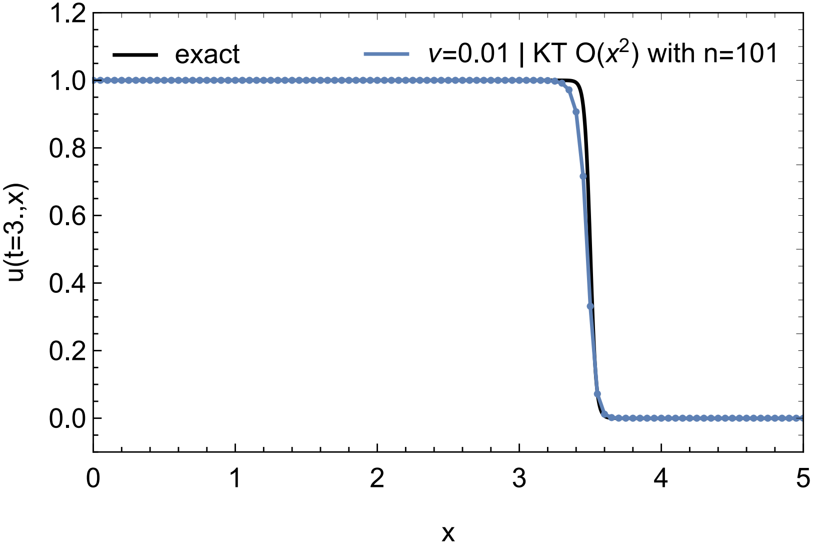
Burgers' eq. (B3)



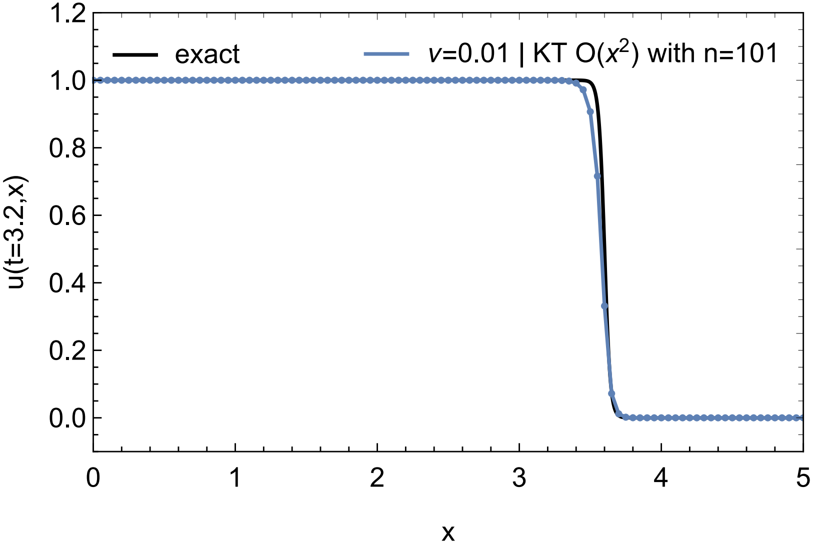


exact
v=0.01 | KT O(x^2) with n=101
1.2
1.0
0.8
0.6
0.4
0.2
0.0
u(t=2.8,x)
0
1
2
3
4
5
x

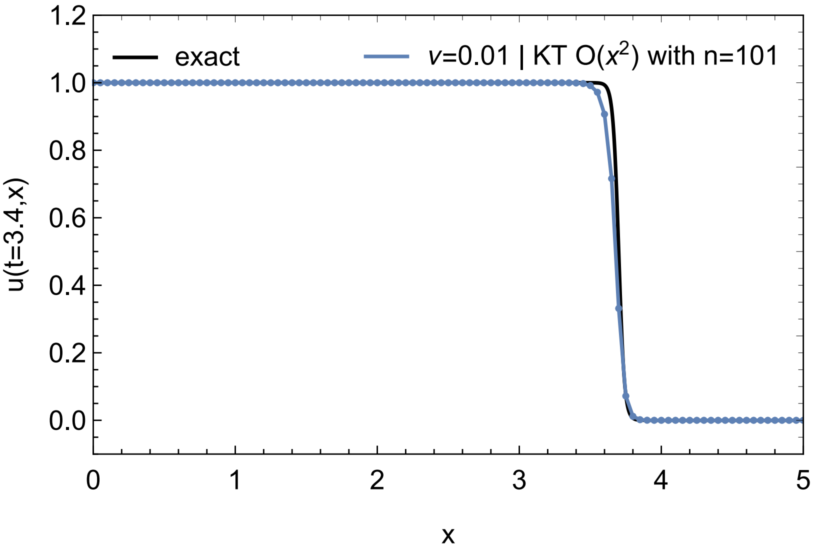
Burgers' eq. (B3)



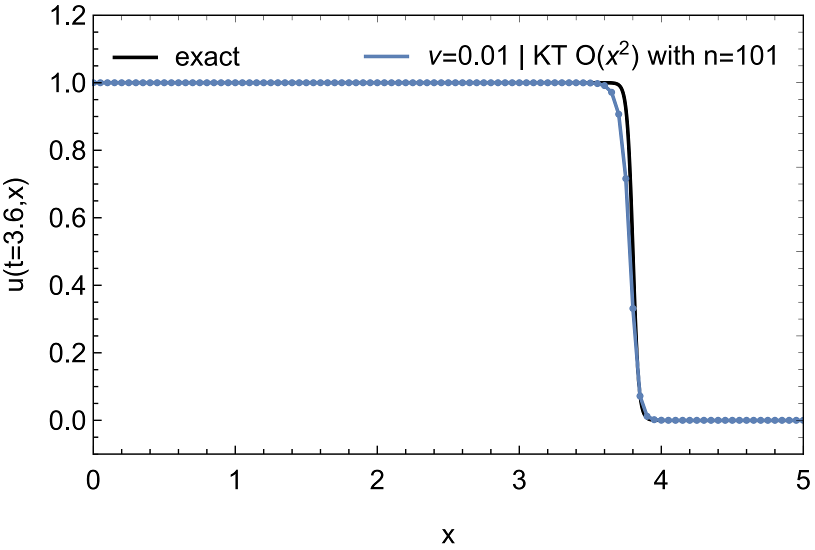
Burgers' eq. (B3)



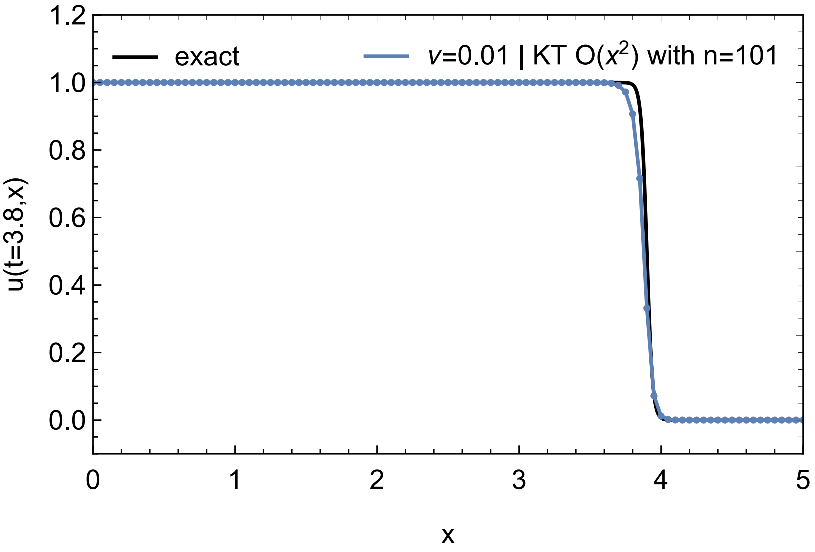
Burgers' eq. (B3)



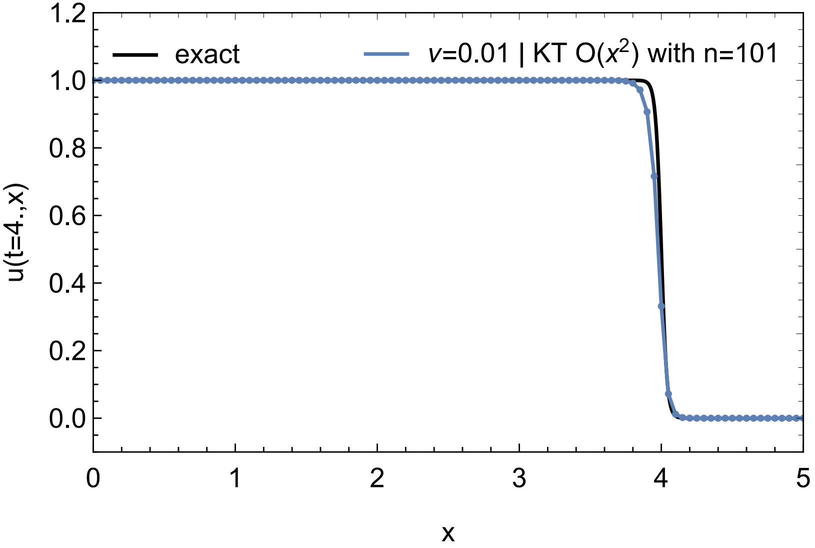
Burgers' eq. (B3)



Burgers' eq. (B3)



Burgers' eq. (B3)



Euler Equations of Gas Dynamics

One-dimensional Euler System:

$$\partial_t \begin{pmatrix} \rho \\ \mu \\ \epsilon \end{pmatrix} + \partial_x \begin{pmatrix} \mu \\ \rho v^2 + p \\ (\epsilon + p) v \end{pmatrix} \equiv \partial_t u + \partial_x F[u] = 0, \tag{E}$$

with $u = (\rho, \mu, \epsilon)^{\mathrm{T}}$.

- Conserved quantities/densities: mass ρ, momentum $\mu = \rho v$ and energy ϵ
- Equation of state: $p = (\gamma - 1)(\epsilon - \rho v^2/2)$ with $\gamma = 1.4$
- Shock tube ICs:

$$u(0, x) = \begin{cases} (\rho_L, \mu_L, \epsilon_L)^{\mathrm{T}} & x \leq x_0 \\ (\rho_R, \mu_R, \epsilon_R)^{\mathrm{T}} & \text{otherwise} \end{cases} \tag{EST}$$

Shock tube initial conditions

All initial conditions are given in SI units and have $x_0 = 0.5\,\mathrm{m}$:

- Sod's classical initial condition[2]:

$$u_L = (1.0, 0.0, 2.5)^{\mathrm{T}} \qquad u_R = (0.125, 0.0, 0.25)^{\mathrm{T}} \qquad \text{(EST.Sod)}$$

- Very strong shock[3]:

$$u_L = (0.445, 0.311, 8.928)^{\mathrm{T}} \qquad u_R = (0.5, 0.0, 1.4275)^{\mathrm{T}} \qquad \text{(EST.Lax)}$$

- Strong double rarefaction wave and vacuum limit[4]:

$$u_L = (1.0, -2.0, 3.0)^{\mathrm{T}} \qquad u_R = (1.0, 2.0, 3.0)^{\mathrm{T}} \qquad \text{(EST.Toro2)}$$

Exact solutions of the respective Riemann problems were computed with[5]

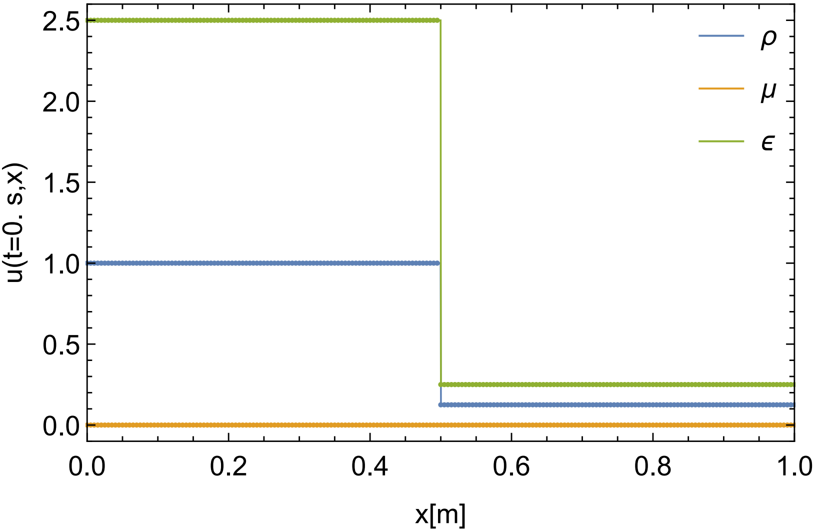
[2] G. A. Sod, J. Comput. Phys. **27**.1 (1978)

[3] P. D. Lax, Comm. Pure Appl. Math. **7**.1 (1954)

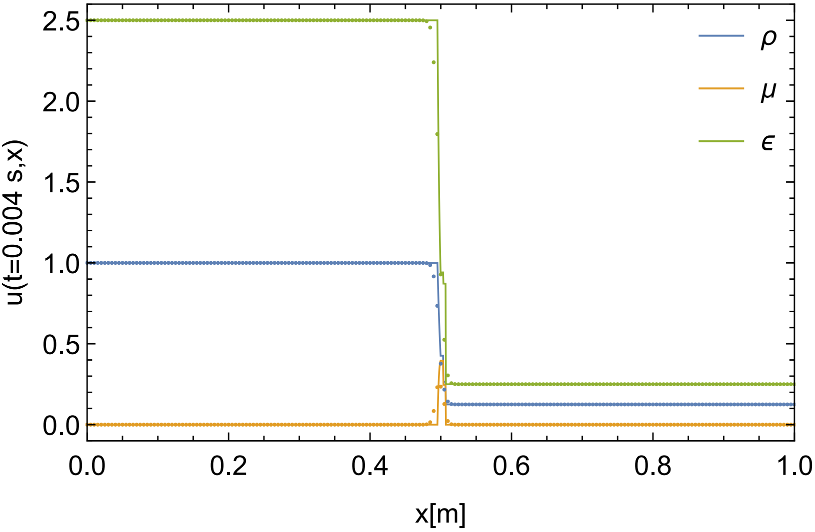
[4] E. Toro, *Riemann Solvers and Numerical Methods for Fluid Dynamics*, Berlin: Springer, 1999

[5] D. I. Ketcheson and et al., Riemann Problems and Jupyter Solutions, github.com/clawpack/riemann_book - 2019.07.09 - 20:30

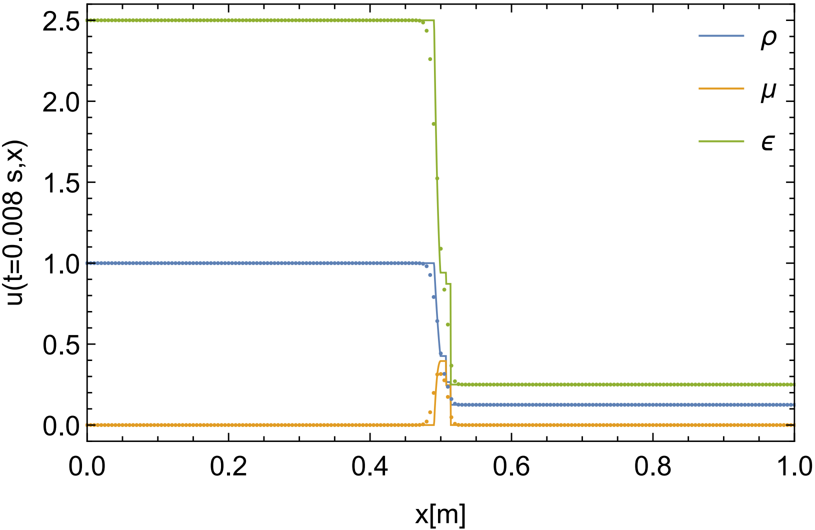
Euler shock tube (EST.Sod) - KT $O(x^2)$ n=201



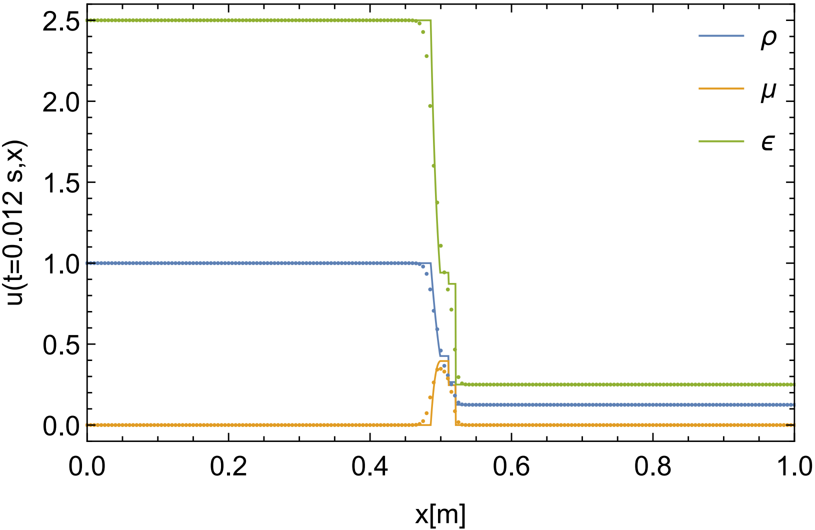
Euler shock tube (EST.Sod) - KT $O(x^2)$ n=201



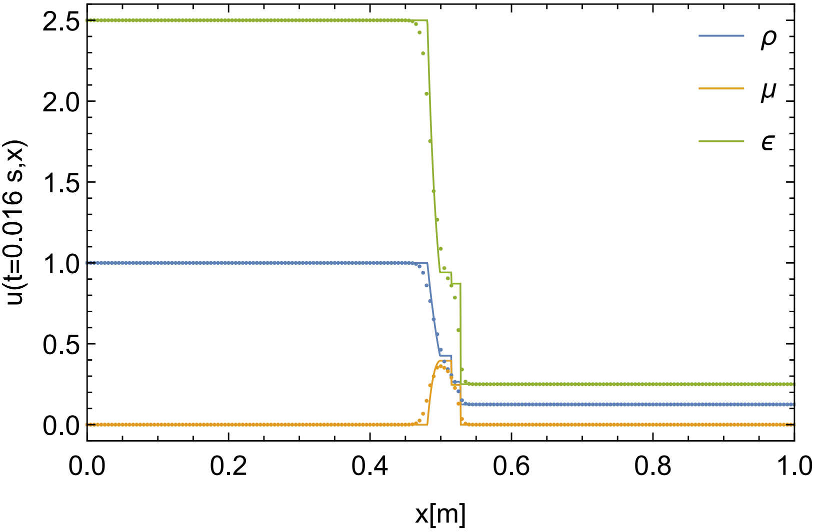
Euler shock tube (EST.Sod) - KT $O(x^2)$ n=201



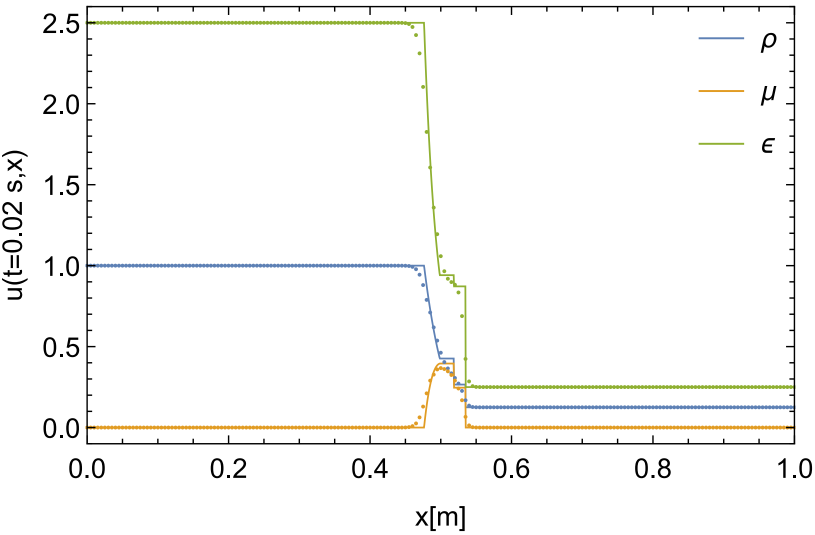
Euler shock tube (EST.Sod) - KT $O(x^2)$ n=201



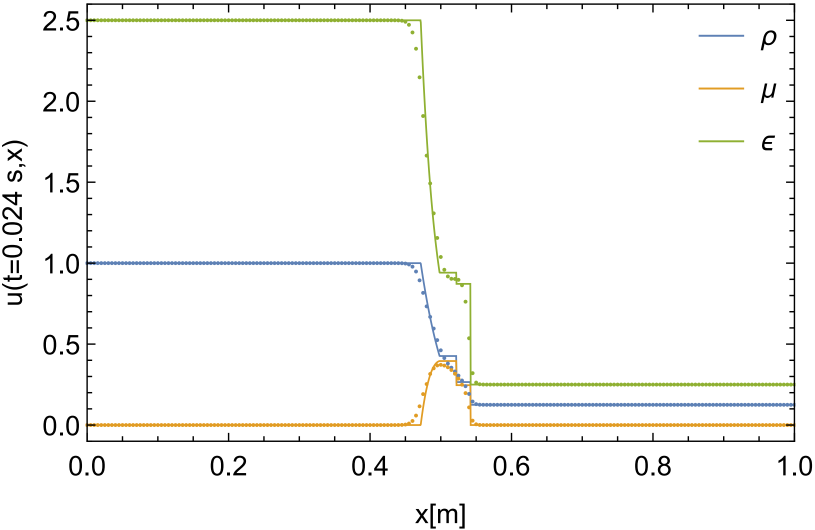
Euler shock tube (EST.Sod) - KT $O(x^2)$ n=201



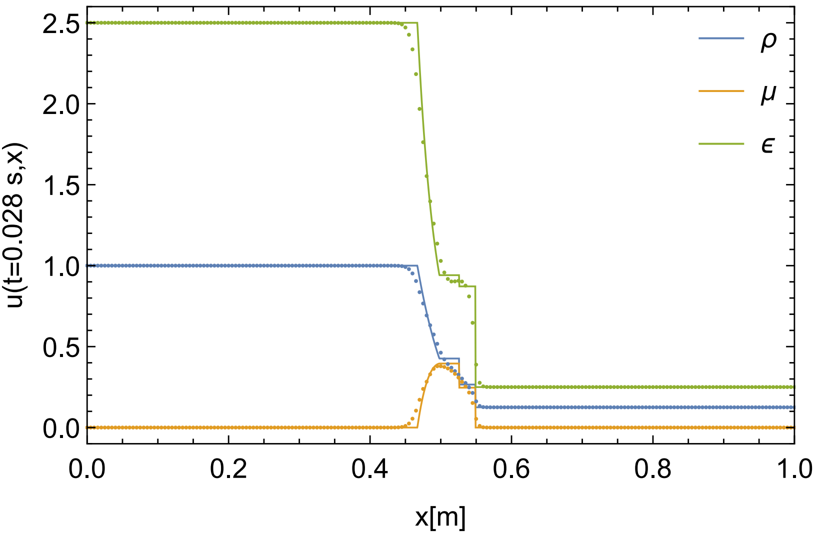
Euler shock tube (EST.Sod) - KT $O(x^2)$ n=201



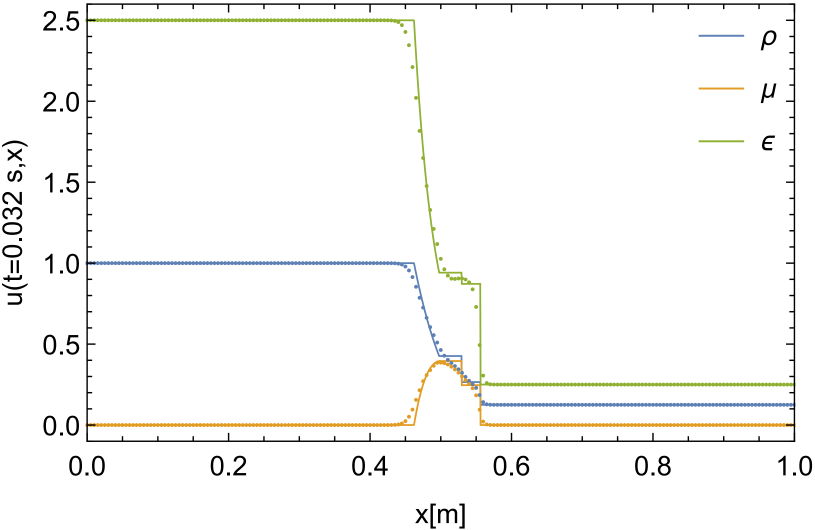
Euler shock tube (EST.Sod) - KT $O(x^2)$ n=201

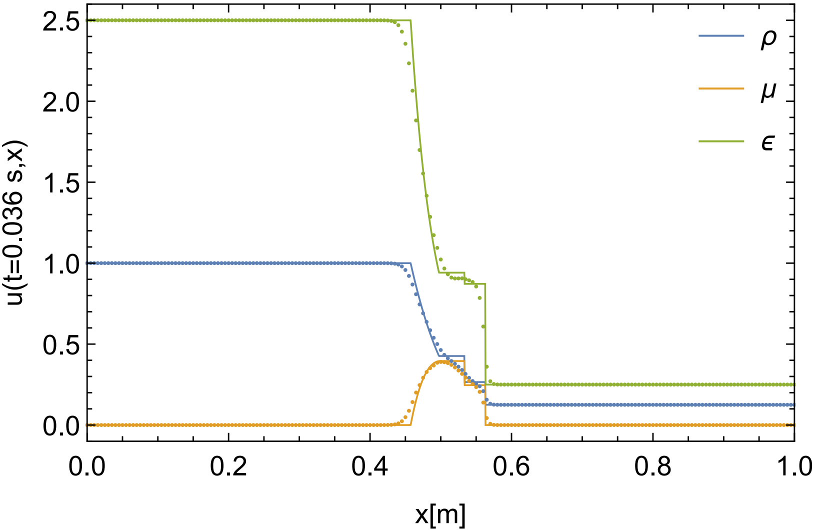


Euler shock tube (EST.Sod) - KT $O(x^2)$ n=201

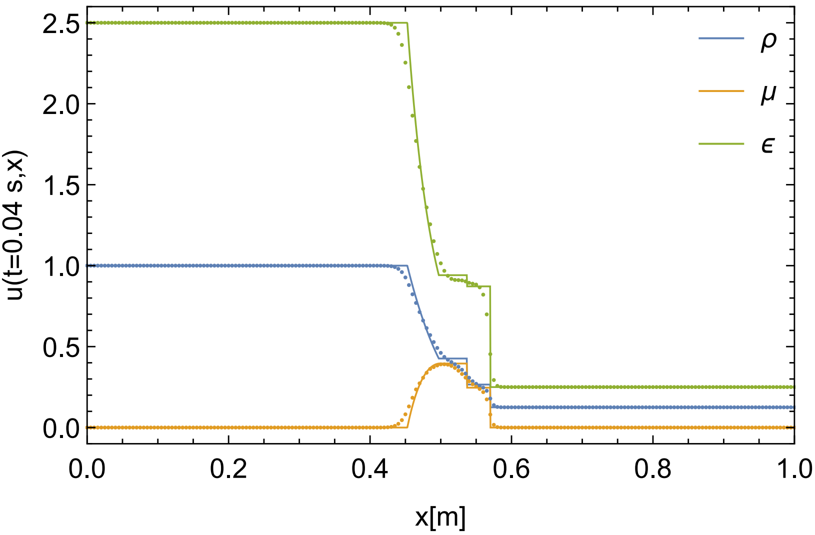


Euler shock tube (EST.Sod) - KT $O(x^2)$ n=201

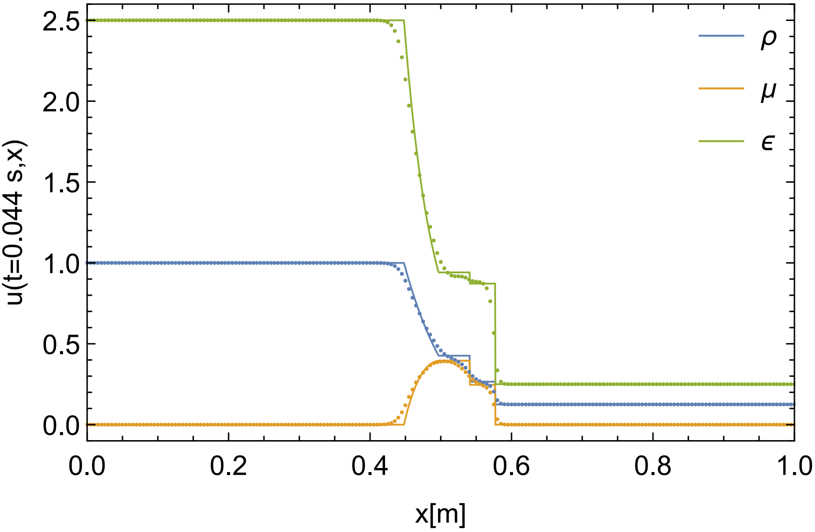



2.5
2.0
1.5
1.0
0.5
0.0
u(t=0.036 s,x)
0.0
0.2
0.4
0.6
0.8
1.0
x[m]
ρ
μ
ϵ

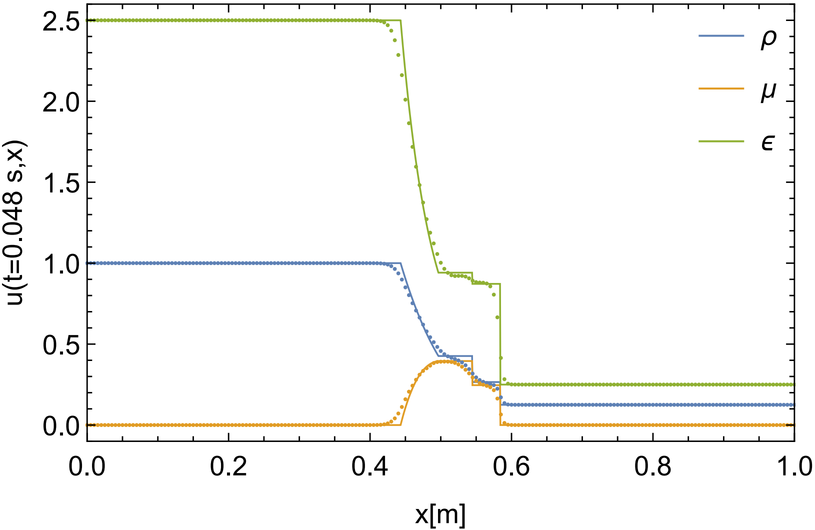
Euler shock tube (EST.Sod) - KT $O(x^2)$ n=201



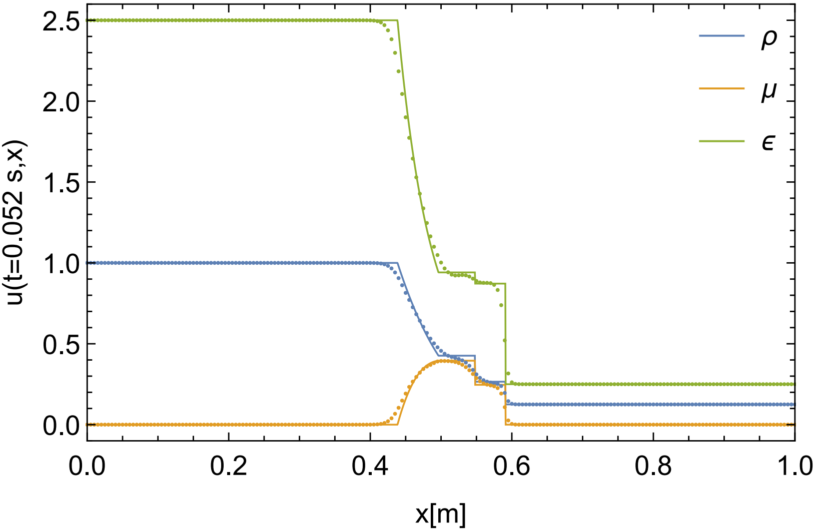
Euler shock tube (EST.Sod) - KT $O(x^2)$ n=201



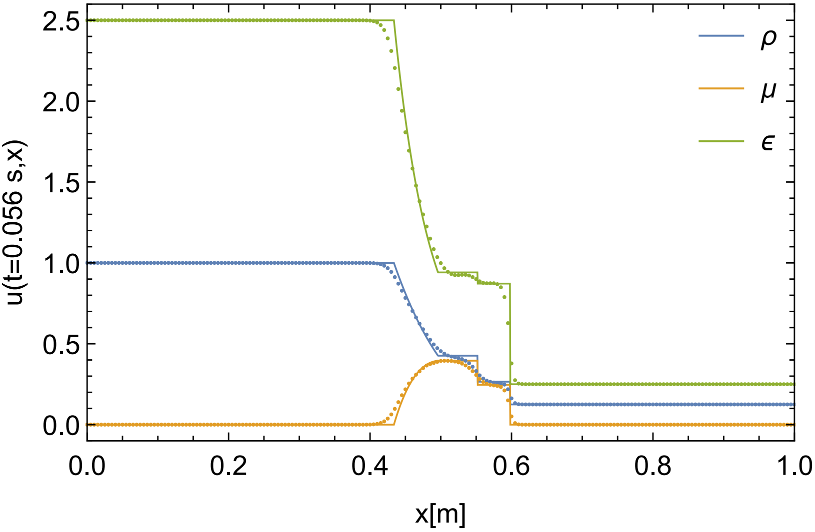
Euler shock tube (EST.Sod) - KT $O(x^2)$ n=201



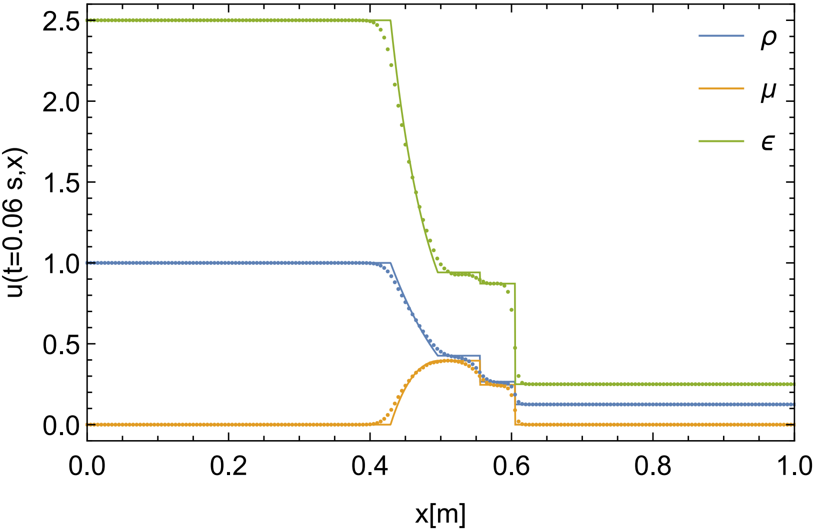
Euler shock tube (EST.Sod) - KT $O(x^2)$ n=201



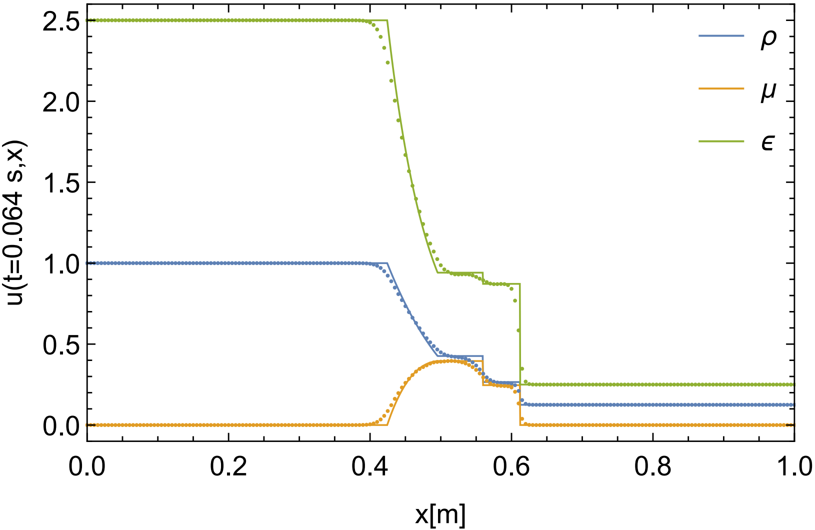
Euler shock tube (EST.Sod) - KT $O(x^2)$ n=201

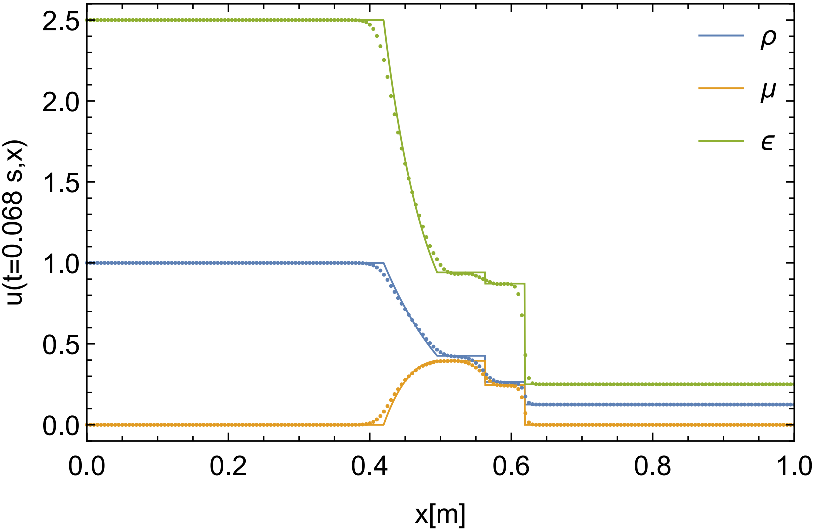


Euler shock tube (EST.Sod) - KT $O(x^2)$ n=201



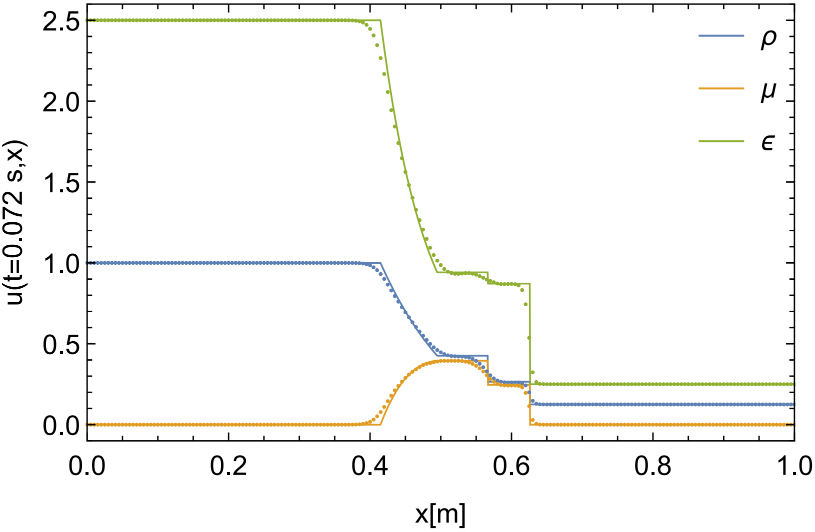
Euler shock tube (EST.Sod) - KT $O(x^2)$ n=201



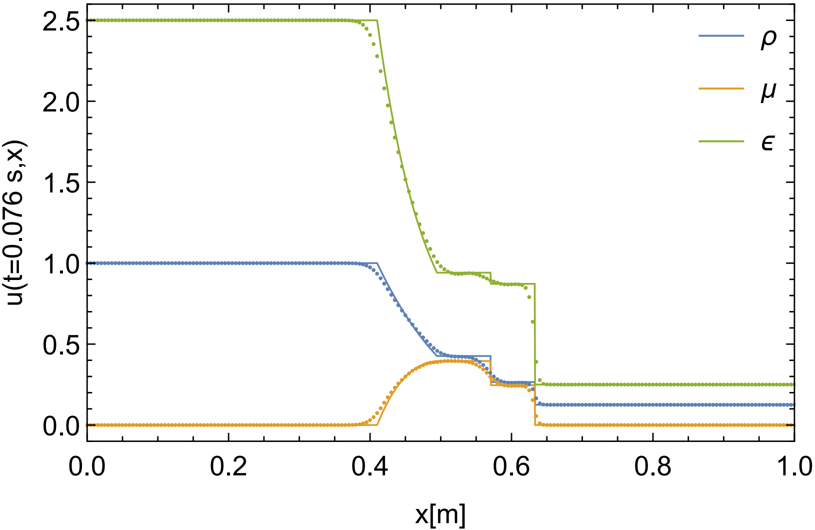


2.5
2.0
1.5
1.0
0.5
0.0
u(t=0.068 s,x)
0.0
0.2
0.4
0.6
0.8
1.0
x[m]
ρ
μ
ϵ

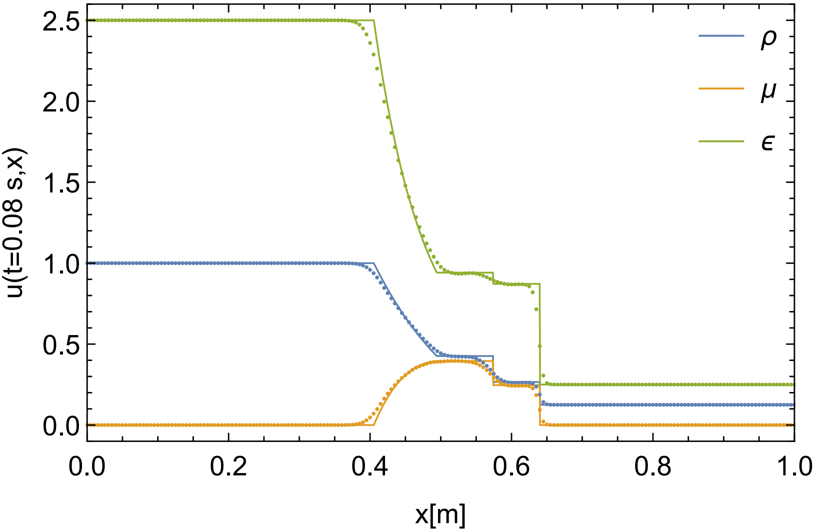
Euler shock tube (EST.Sod) - KT $O(x^2)$ n=201



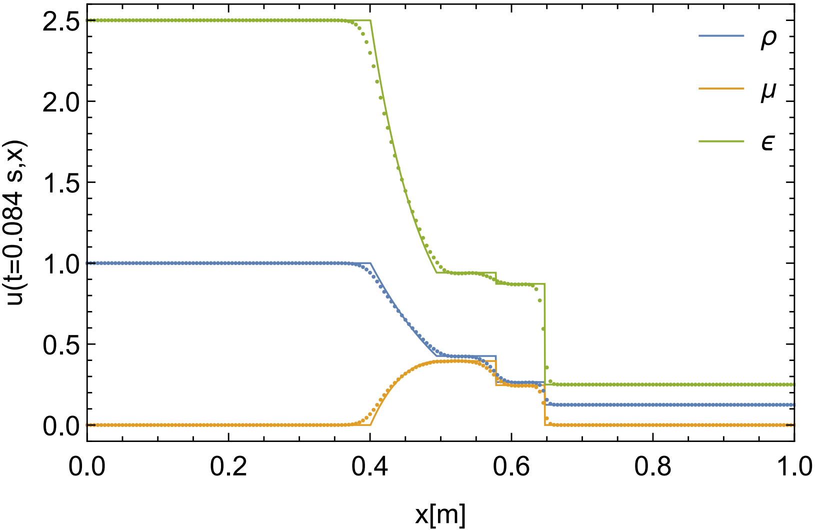
Euler shock tube (EST.Sod) - KT $O(x^2)$ n=201



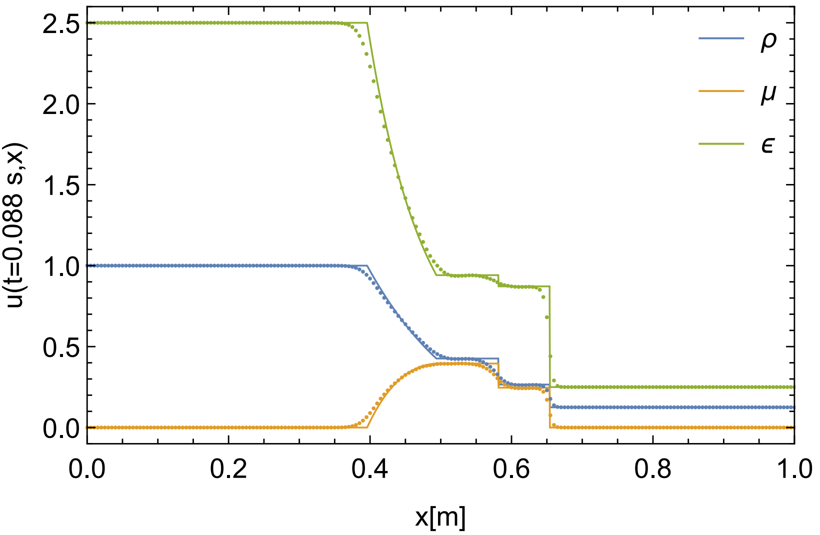
Euler shock tube (EST.Sod) - KT $O(x^2)$ n=201



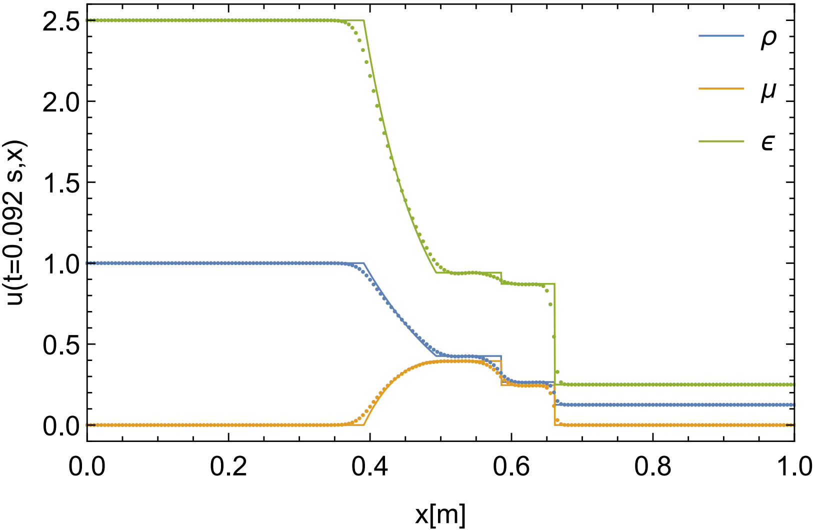
Euler shock tube (EST.Sod) - KT $O(x^2)$ n=201



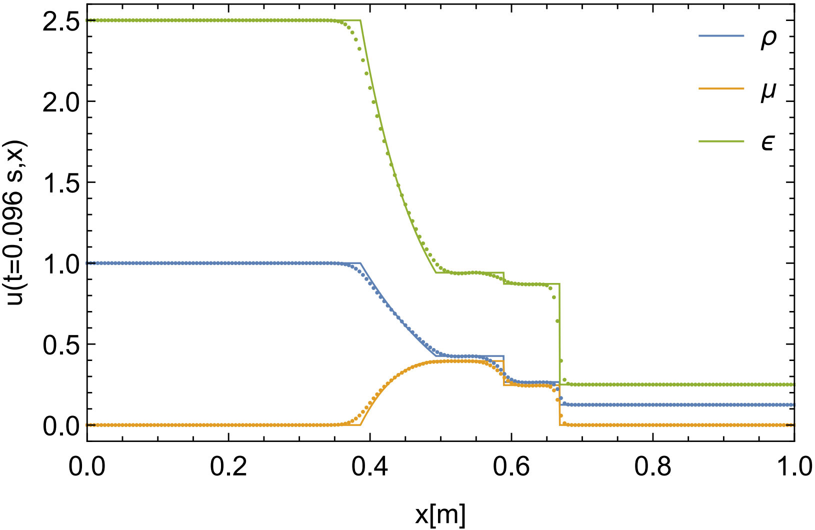
Euler shock tube (EST.Sod) - KT $O(x^2)$ n=201



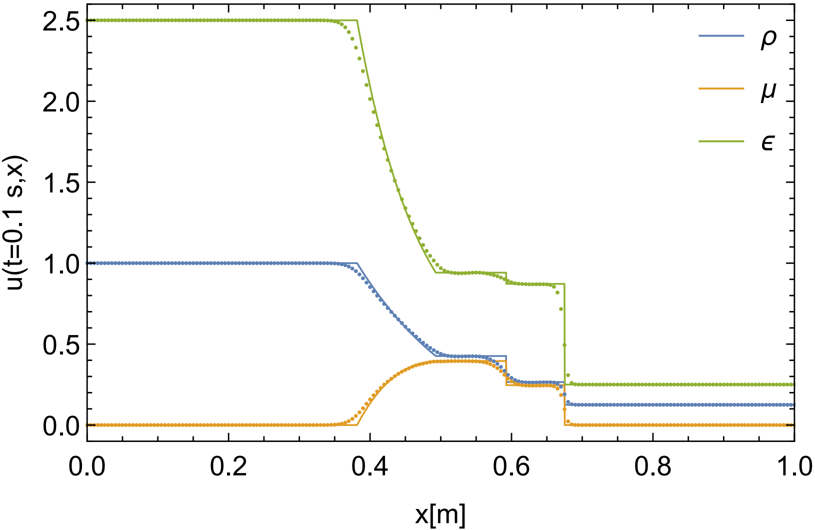
Euler shock tube (EST.Sod) - KT $O(x^2)$ n=201



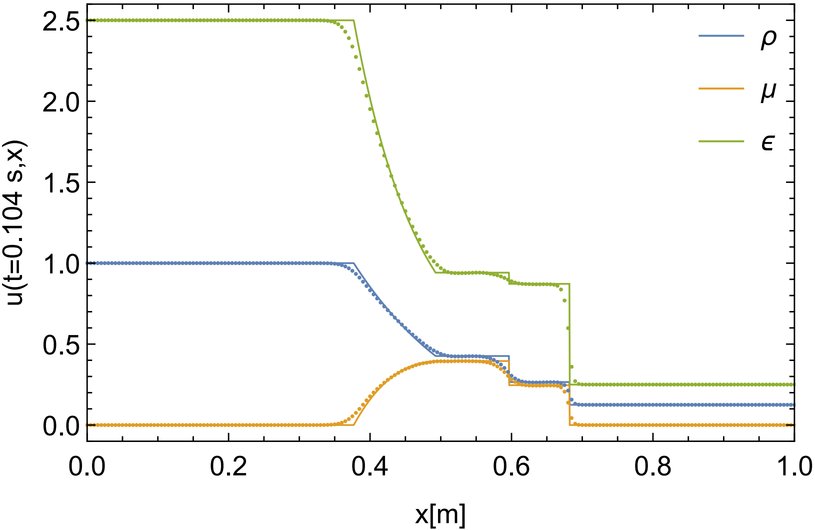
Euler shock tube (EST.Sod) - KT $O(x^2)$ n=201



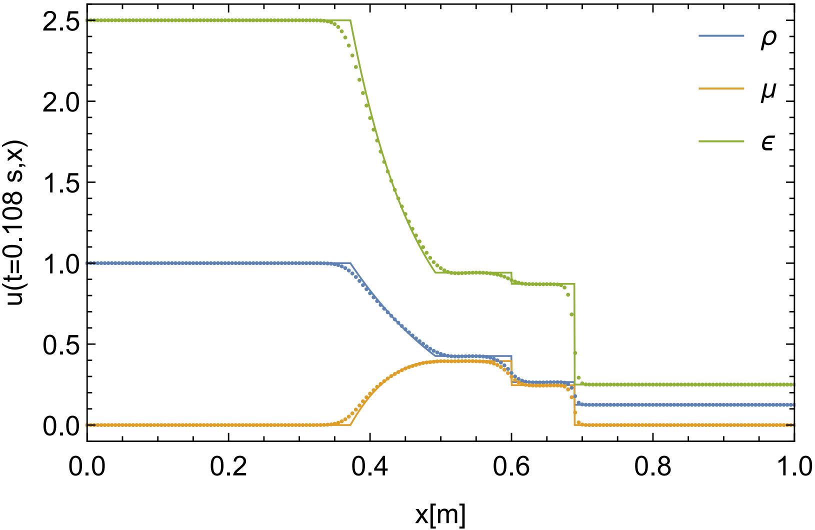
Euler shock tube (EST.Sod) - KT $O(x^2)$ n=201



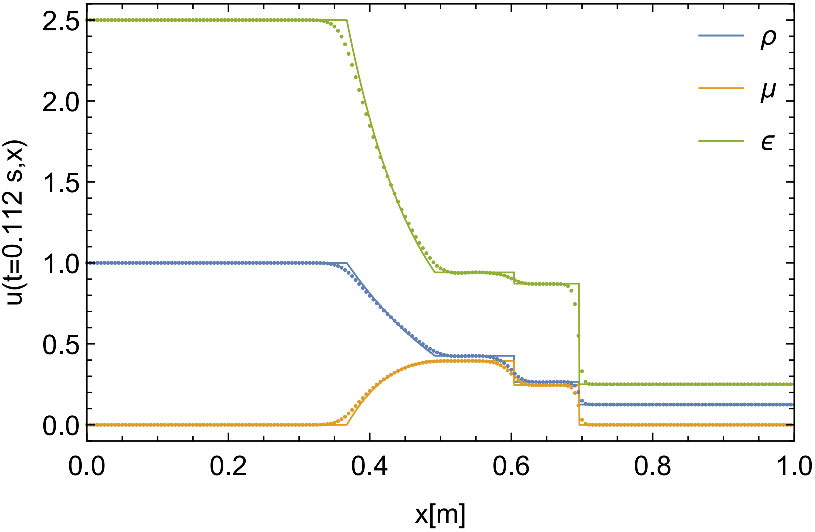
Euler shock tube (EST.Sod) - KT $O(x^2)$ n=201



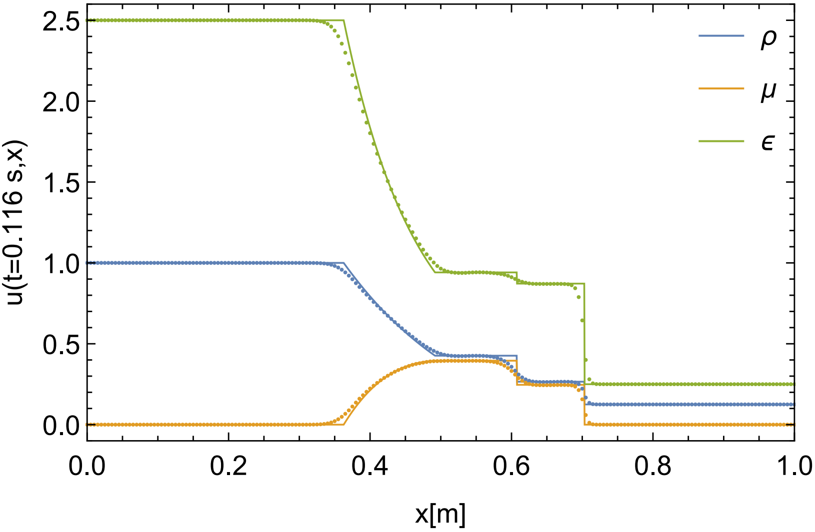
Euler shock tube (EST.Sod) - KT $O(x^2)$ n=201



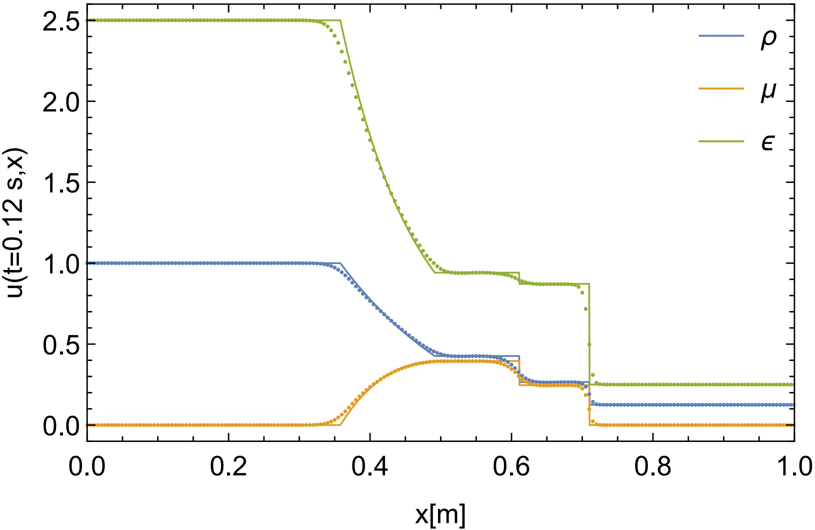
Euler shock tube (EST.Sod) - KT $O(x^2)$ n=201



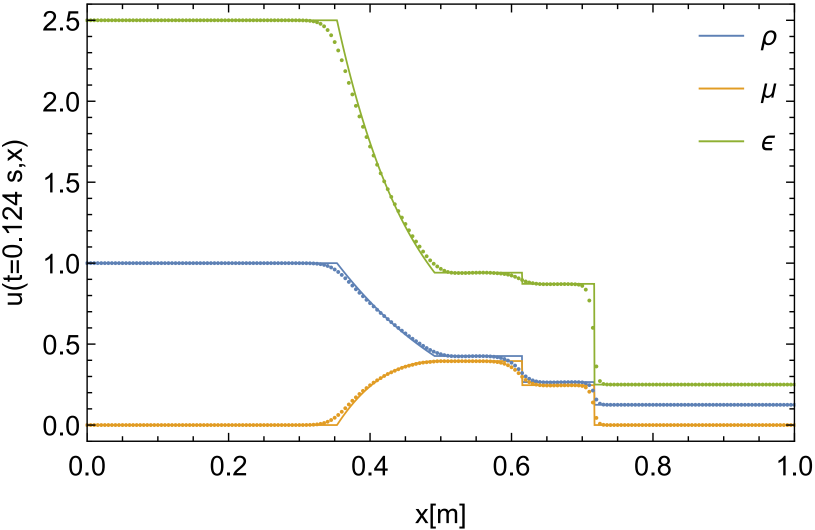
Euler shock tube (EST.Sod) - KT $O(x^2)$ n=201



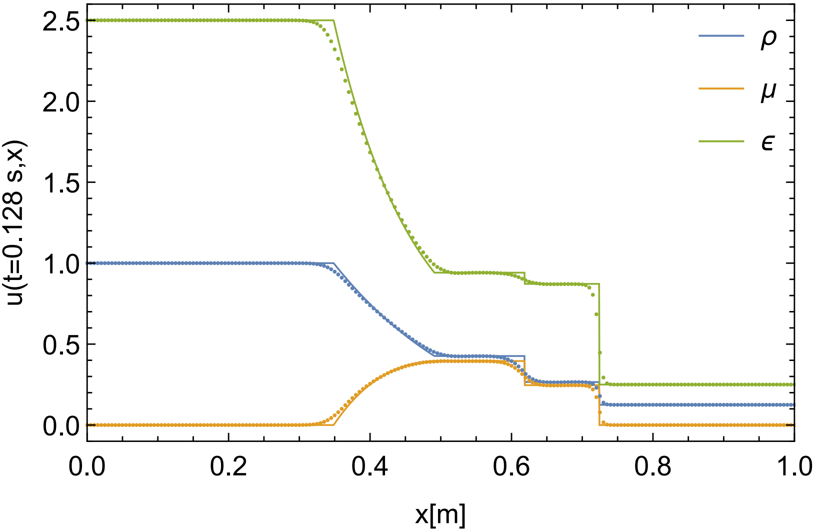
Euler shock tube (EST.Sod) - KT $O(x^2)$ n=201



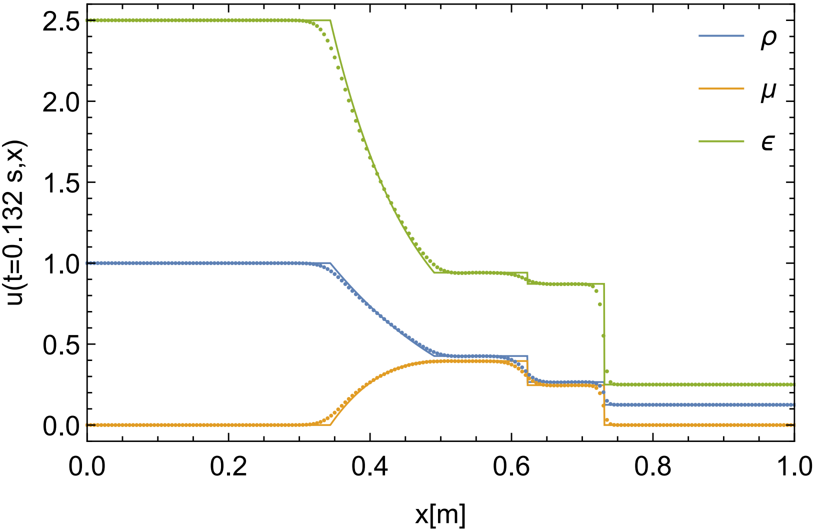
Euler shock tube (EST.Sod) - KT $O(x^2)$ n=201



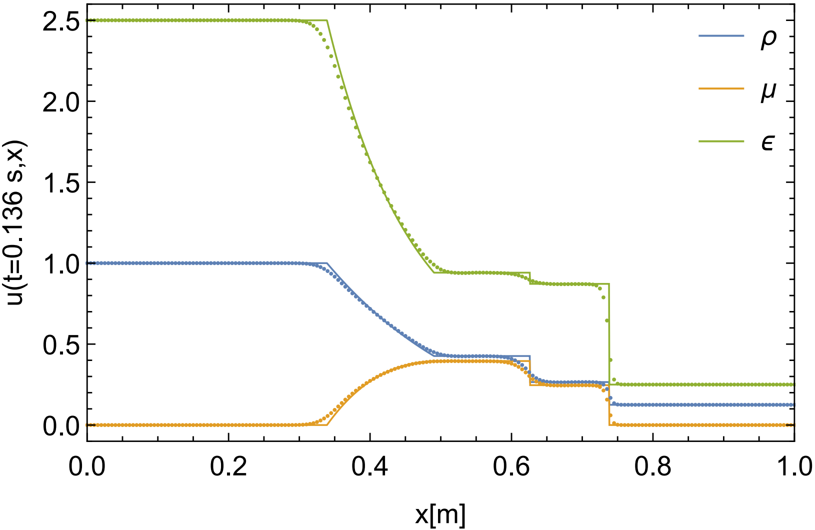
Euler shock tube (EST.Sod) - KT $O(x^2)$ n=201



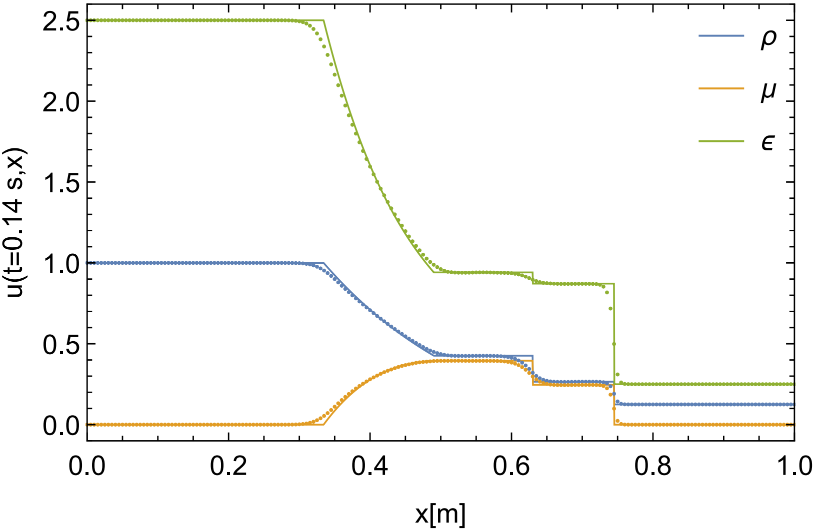
Euler shock tube (EST.Sod) - KT $O(x^2)$ n=201



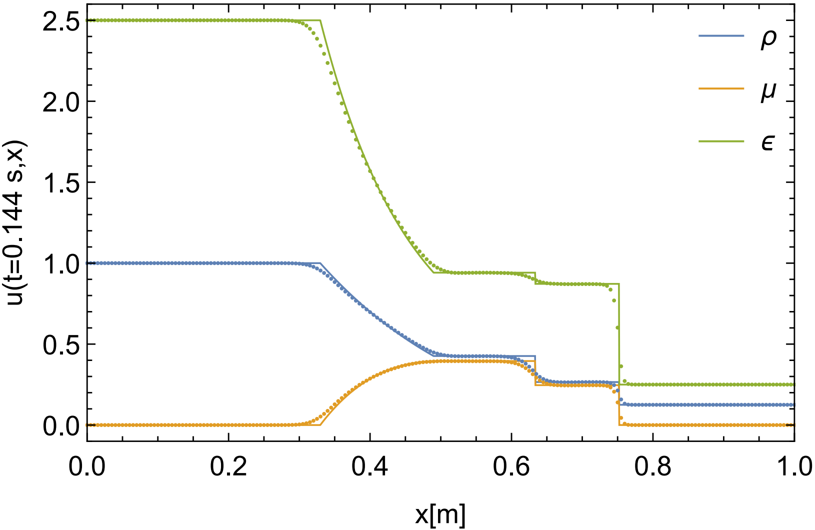
Euler shock tube (EST.Sod) - KT $O(x^2)$ n=201



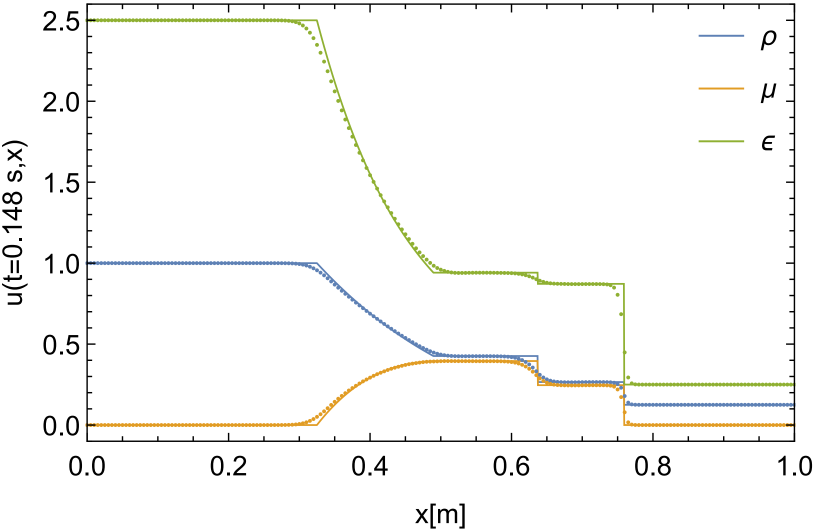
Euler shock tube (EST.Sod) - KT $O(x^2)$ n=201



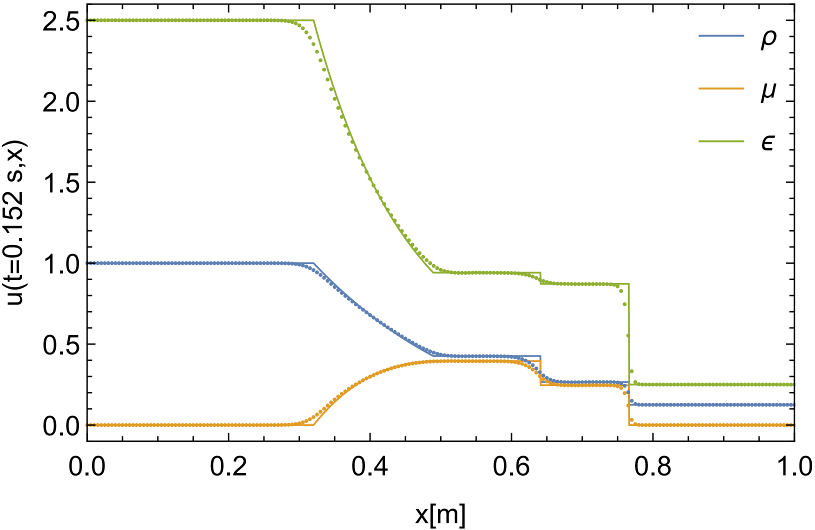
Euler shock tube (EST.Sod) - KT $O(x^2)$ n=201



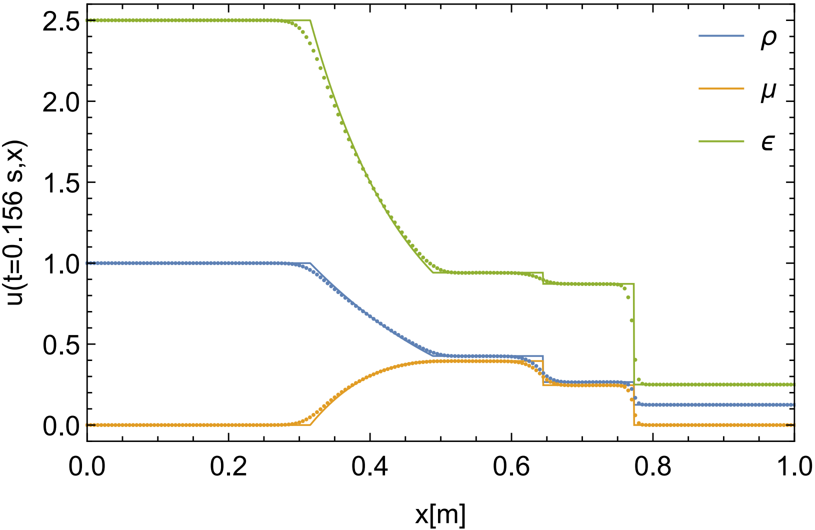
Euler shock tube (EST.Sod) - KT $O(x^2)$ n=201



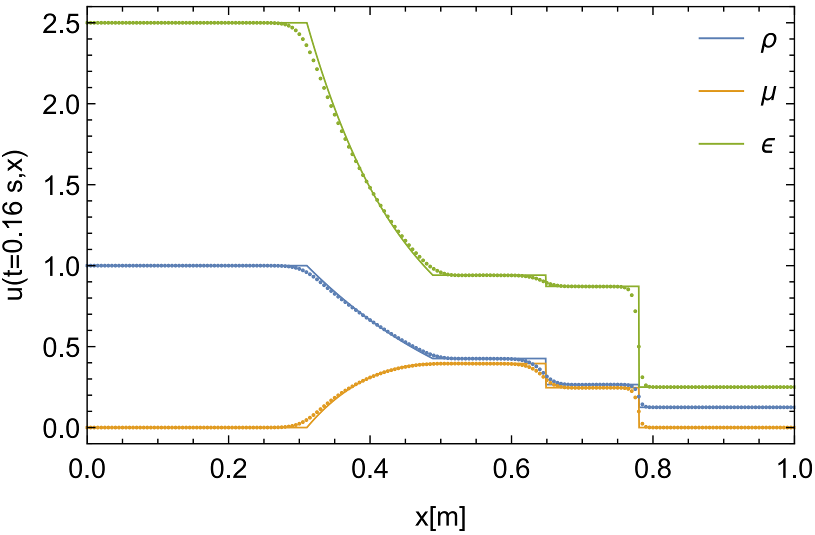
Euler shock tube (EST.Sod) - KT $O(x^2)$ n=201

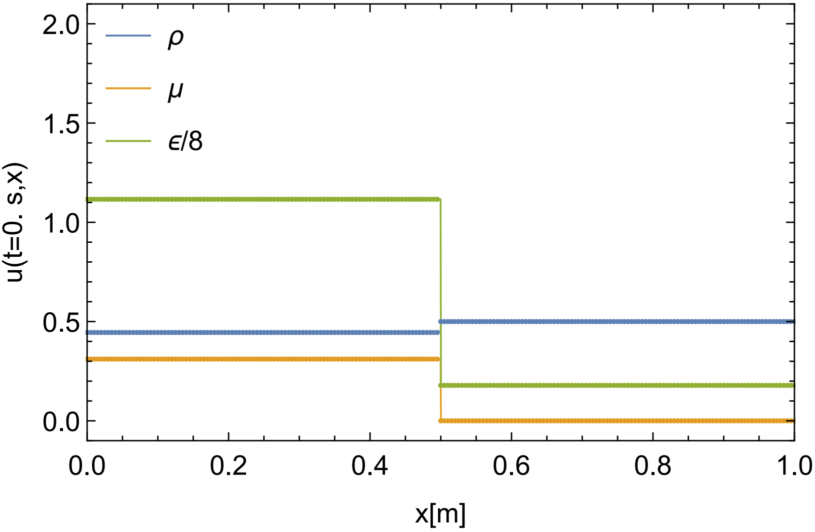


Euler shock tube (EST.Sod) - KT $O(x^2)$ n=201



Euler shock tube (EST.Sod) - KT $O(x^2)$ n=201

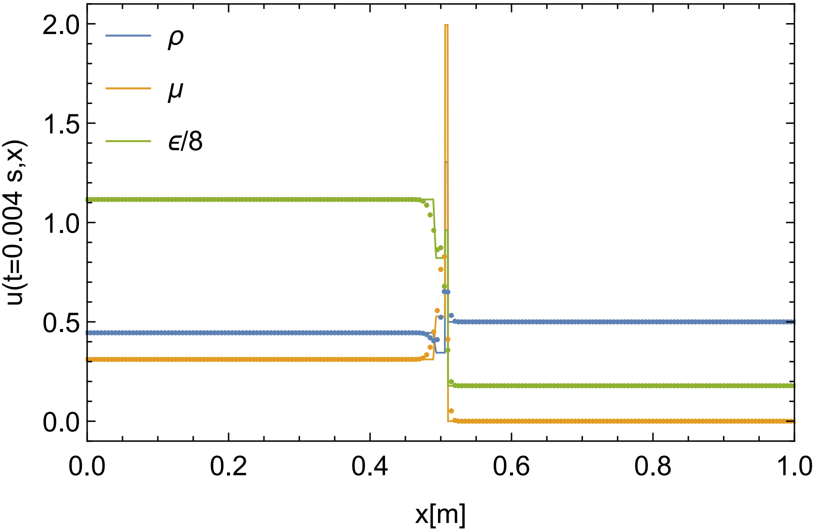


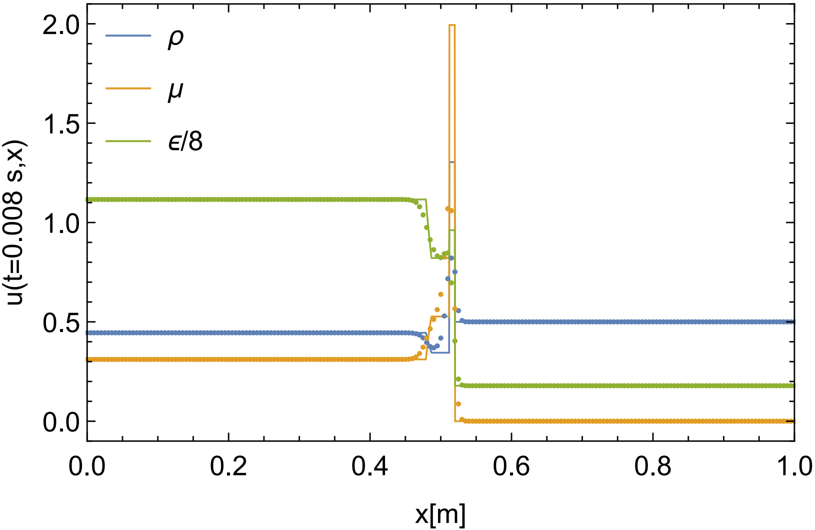


ρ
μ
ϵ/8
2.0
1.5
1.0
0.5
0.0
0.0
0.2
0.4
0.6
0.8
1.0
u(t=0. s,x)
x[m]

Euler shock tube (EST.Lax) - KT $O(x^2)$ n=201

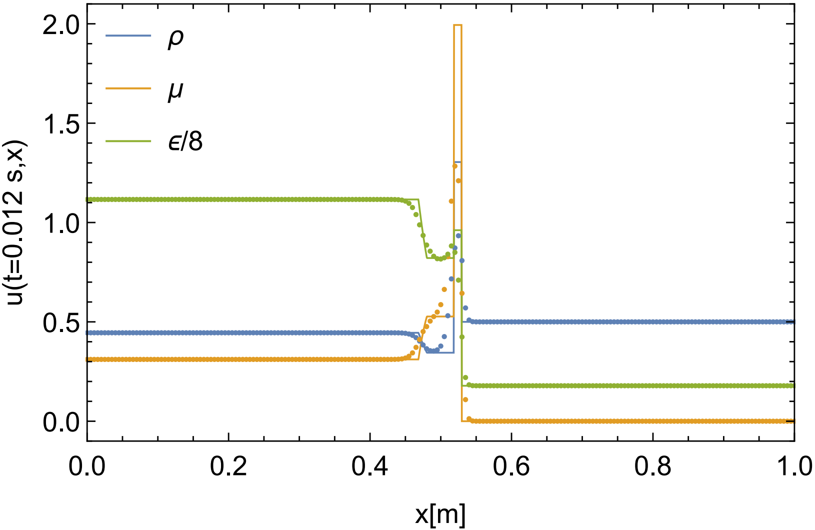
CRC-TR 211



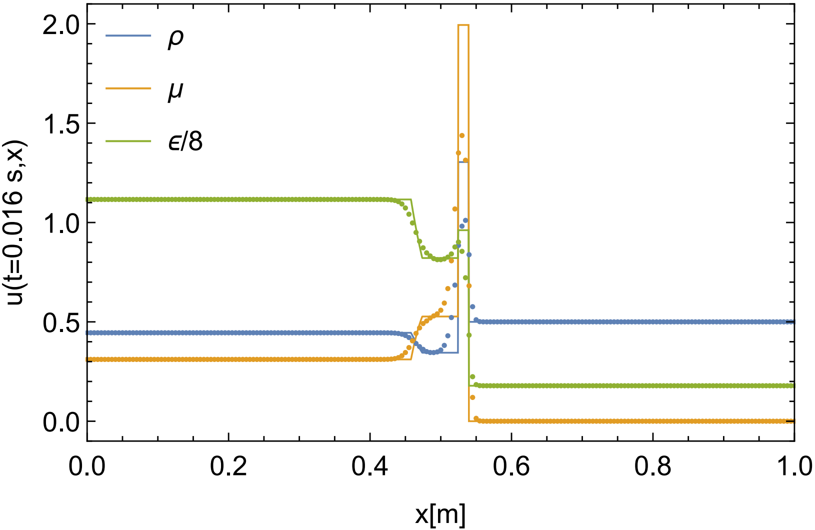


ρ
μ
ϵ/8
u(t=0.008 s,x)
x[m]
0.0
0.5
1.0
1.5
2.0
0.0
0.2
0.4
0.6
0.8
1.0

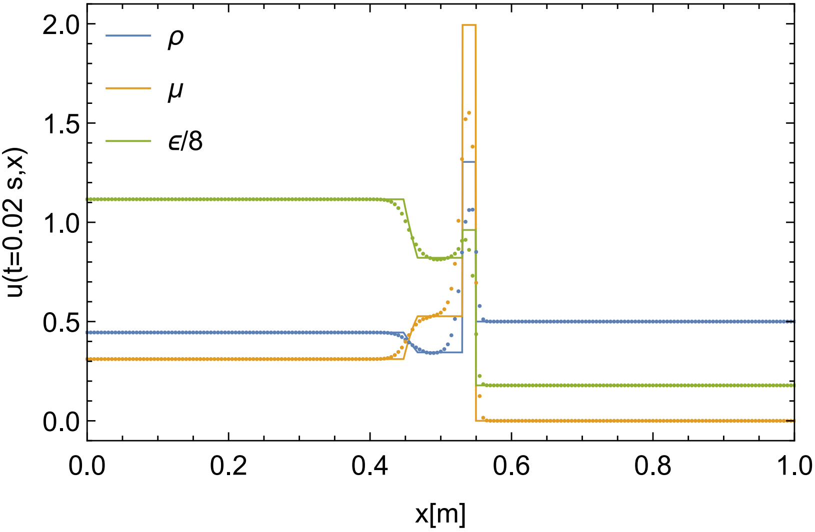
Euler shock tube (EST.Lax) - KT $O(x^2)$ n=201



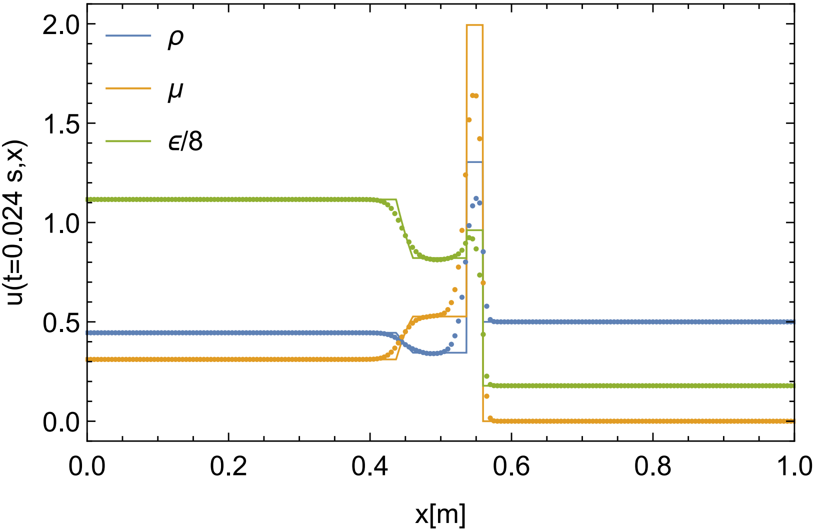
Euler shock tube (EST.Lax) - KT $O(x^2)$ n=201



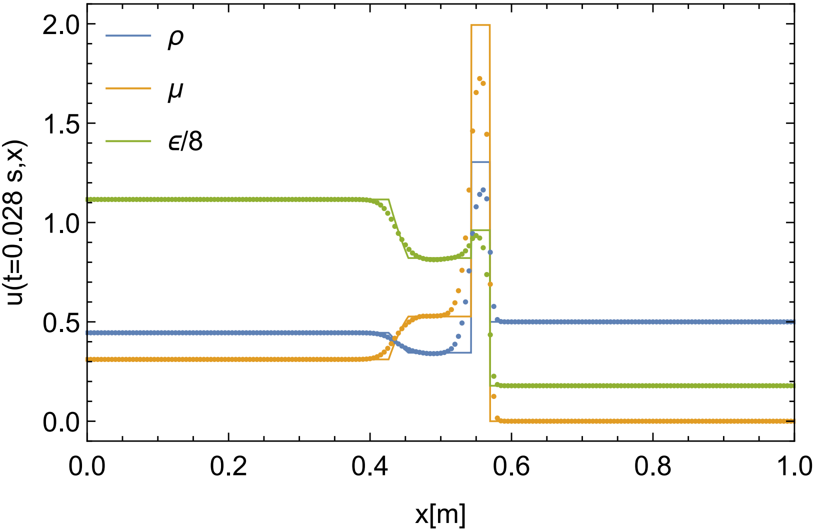
Euler shock tube (EST.Lax) - KT $O(x^2)$ n=201



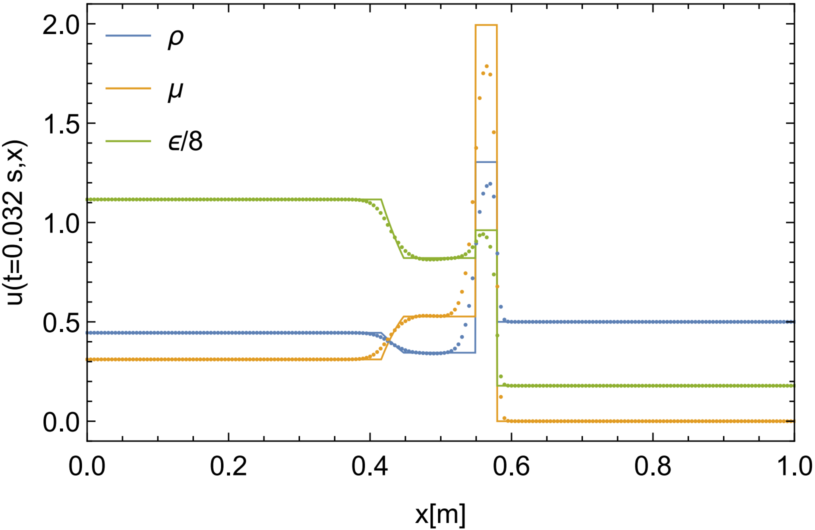
Euler shock tube (EST.Lax) - KT $O(x^2)$ n=201



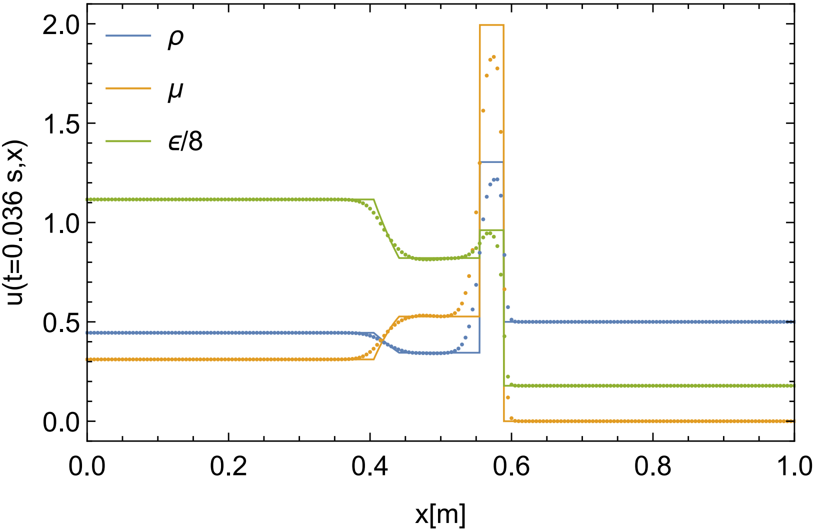
Euler shock tube (EST.Lax) - KT $O(x^2)$ n=201



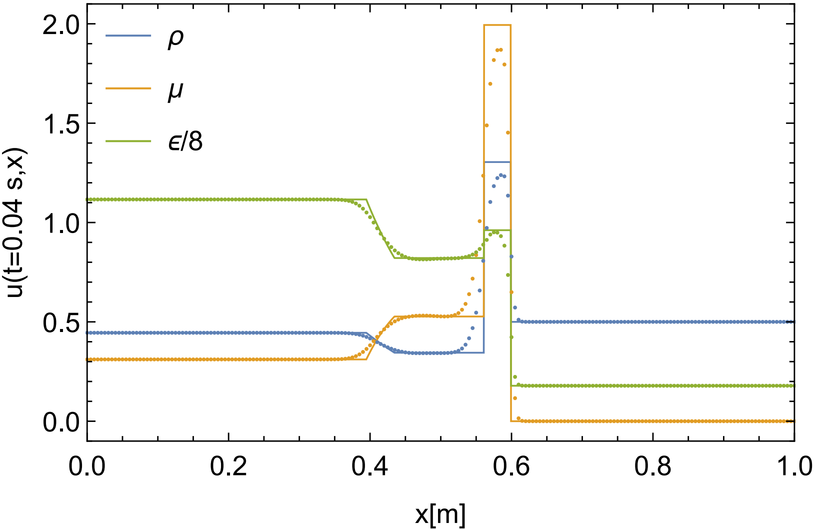
Euler shock tube (EST.Lax) - KT $O(x^2)$ n=201



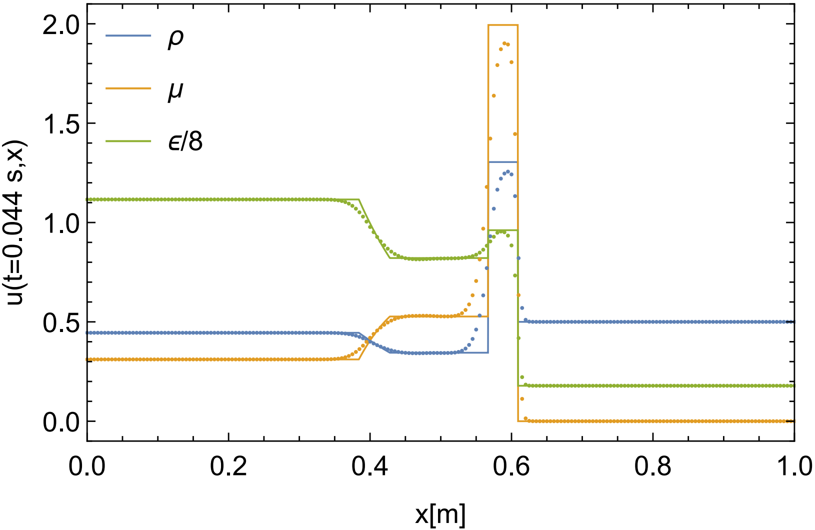
Euler shock tube (EST.Lax) - KT $O(x^2)$ n=201



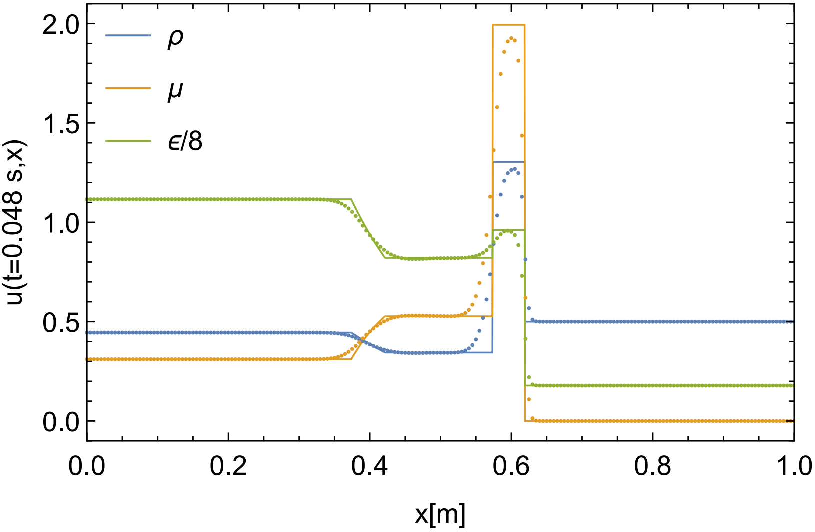
Euler shock tube (EST.Lax) - KT $O(x^2)$ n=201



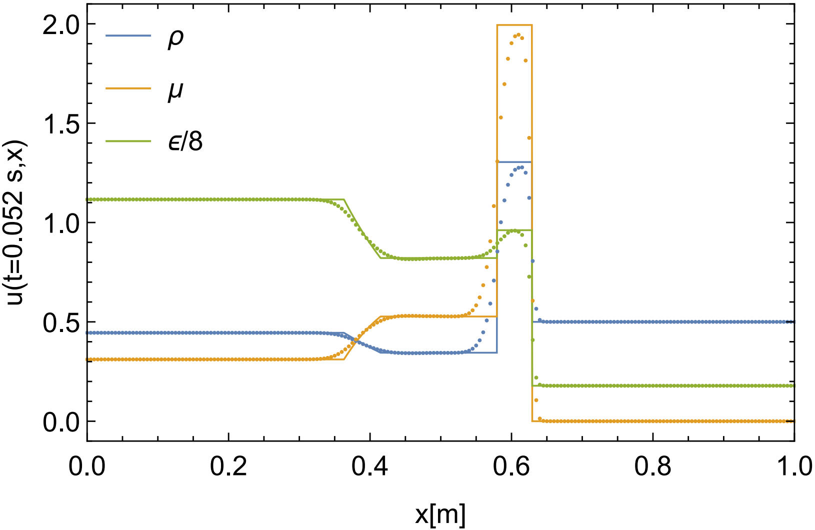
Euler shock tube (EST.Lax) - KT $O(x^2)$ n=201



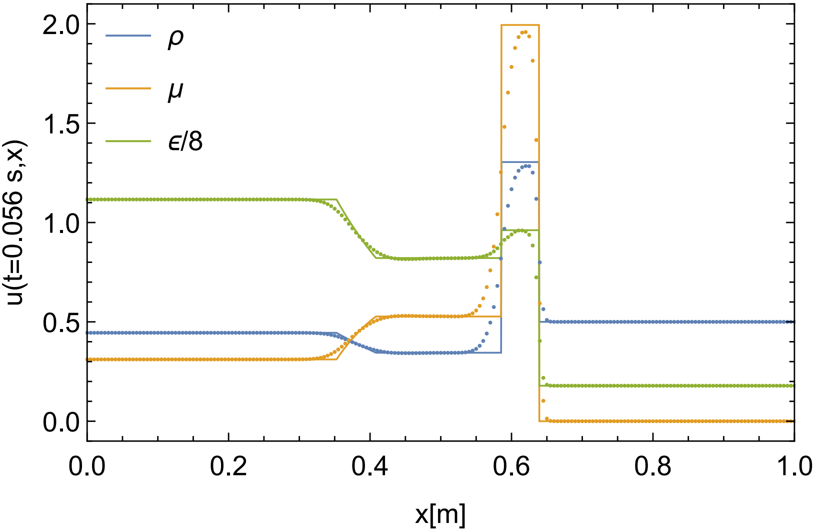
Euler shock tube (EST.Lax) - KT $O(x^2)$ n=201



Euler shock tube (EST.Lax) - KT $O(x^2)$ n=201



Euler shock tube (EST.Lax) - KT $O(x^2)$ n=201



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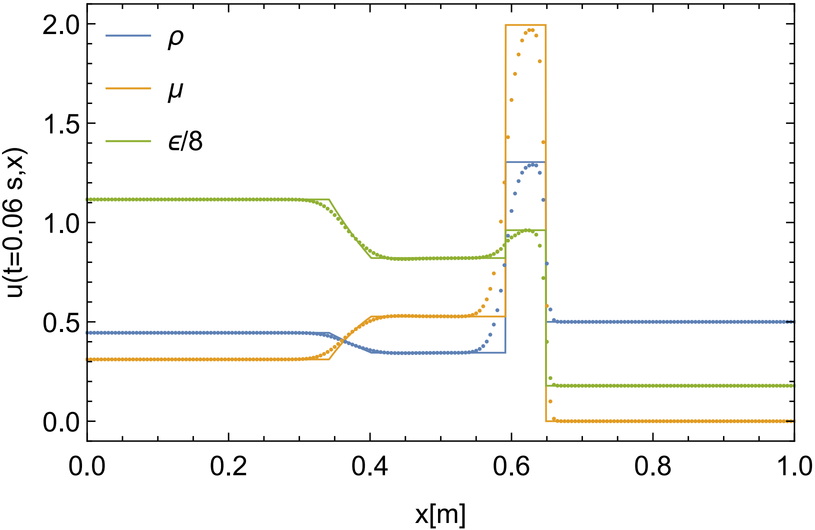
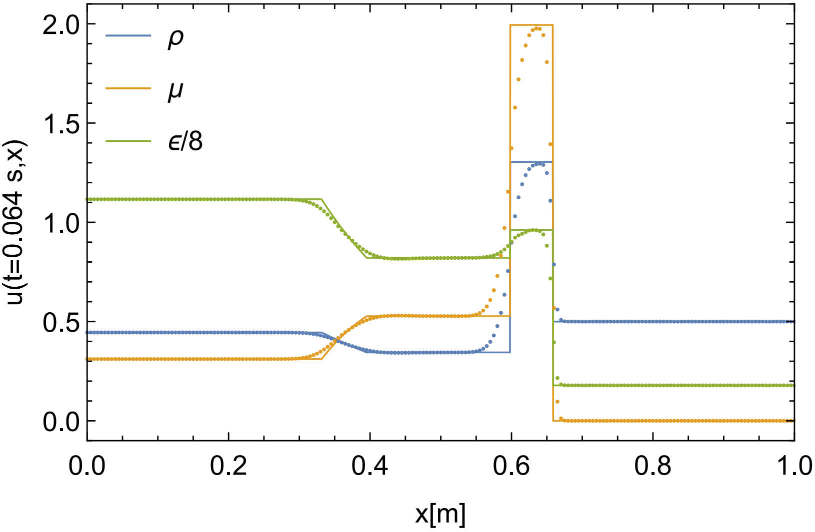
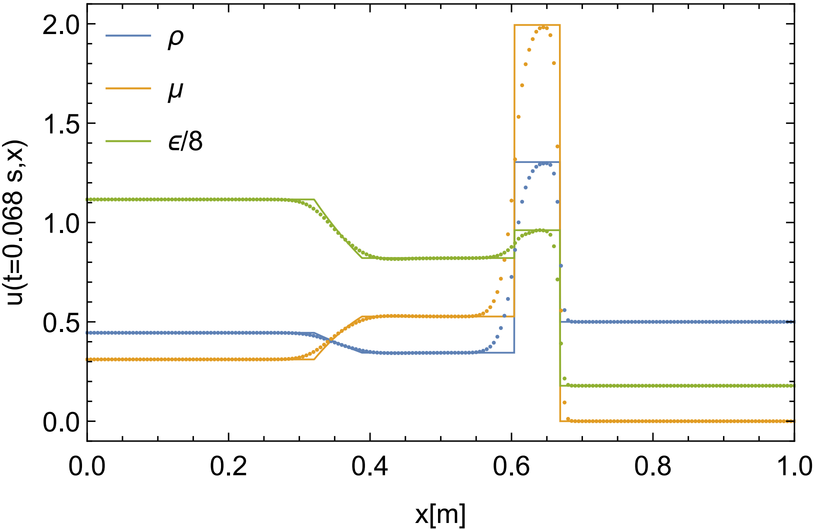


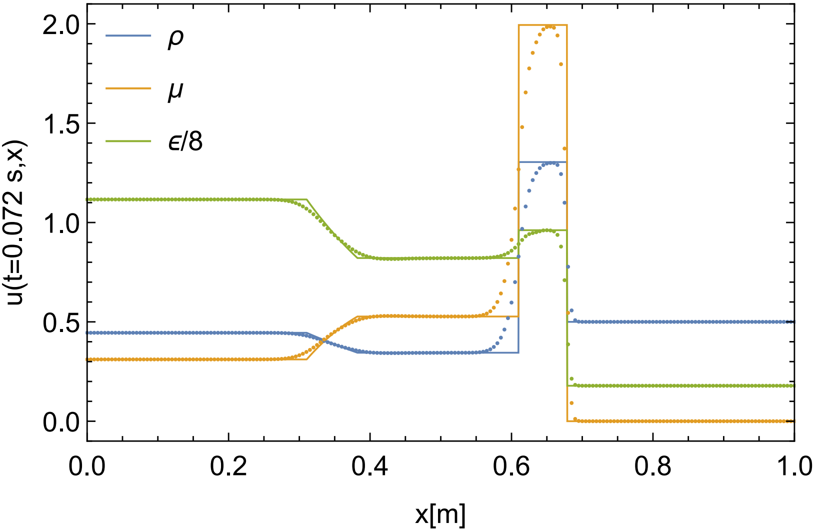
Euler shock tube (EST.Lax) - KT $O(x^2)$ n=201



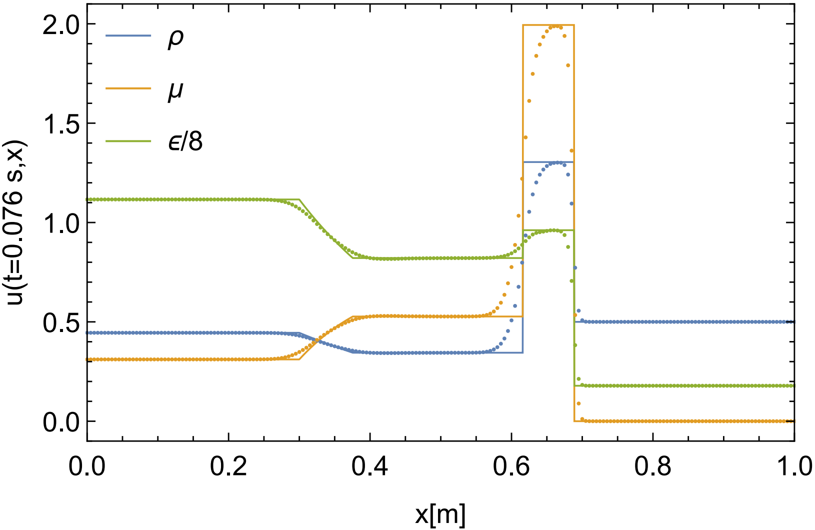
Euler shock tube (EST.Lax) - KT $O(x^2)$ n=201



Euler shock tube (EST.Lax) - KT $O(x^2)$ n=201



Euler shock tube (EST.Lax) - KT $O(x^2)$ n=201



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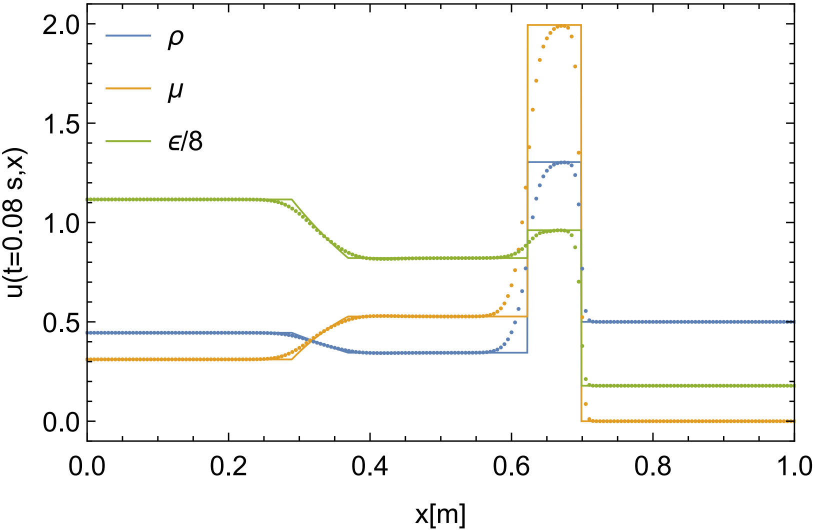

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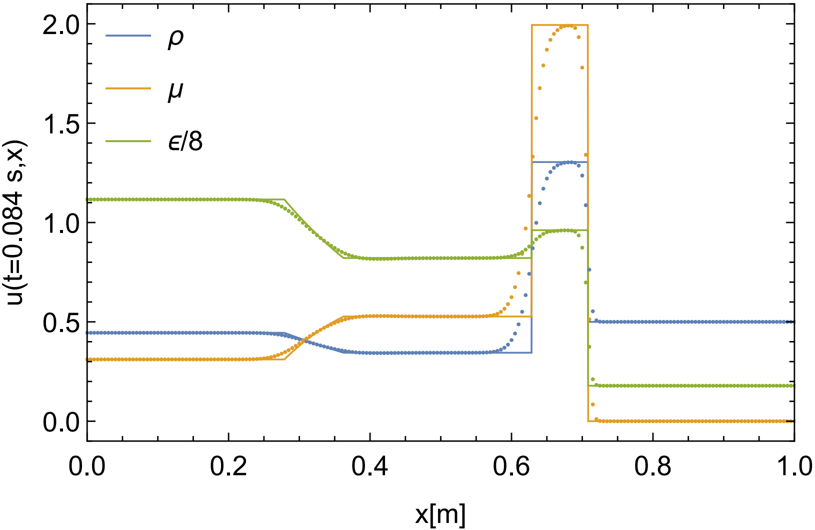

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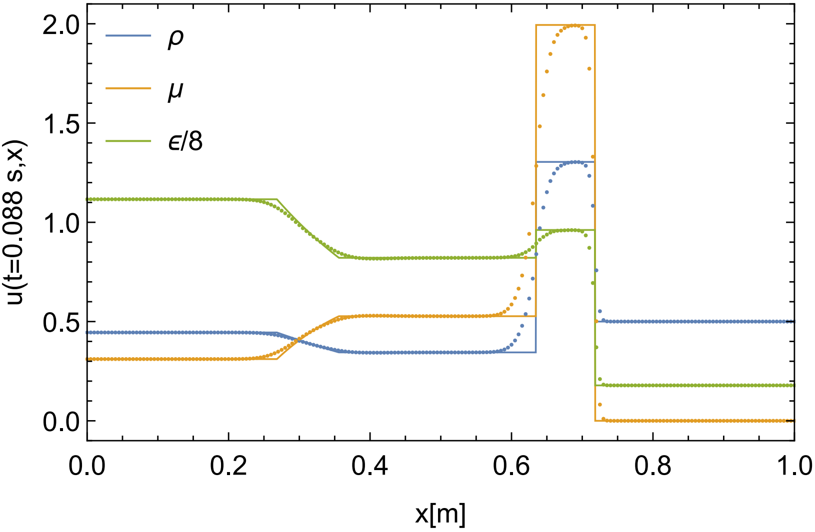
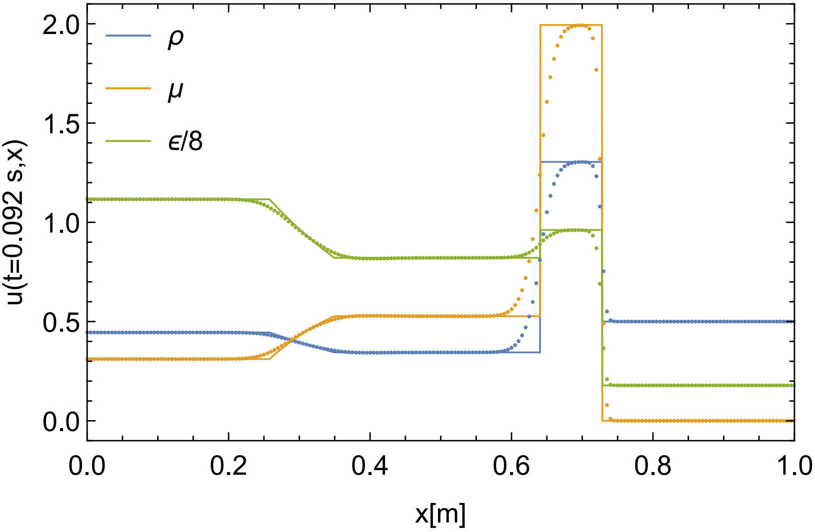
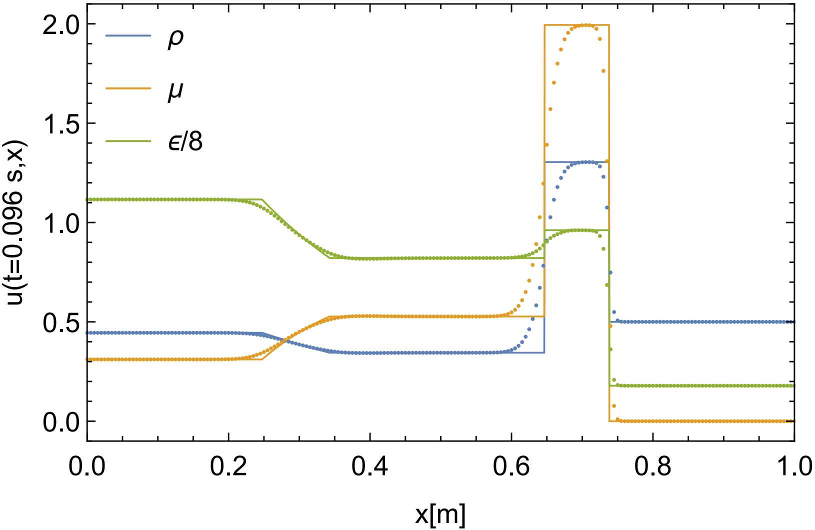


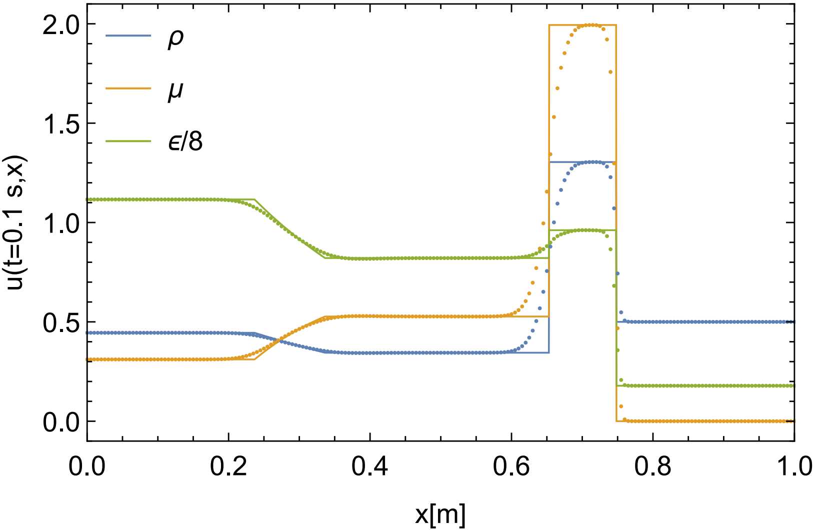
Euler shock tube (EST.Lax) - KT $O(x^2)$ n=201



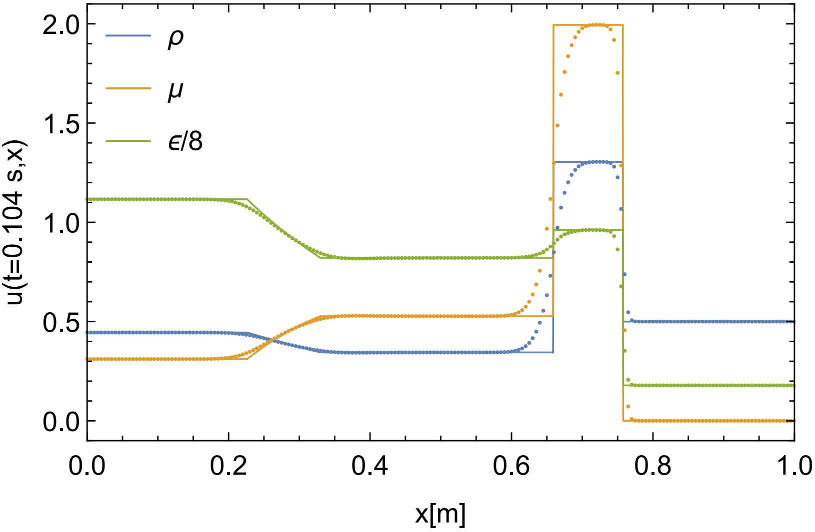
Euler shock tube (EST.Lax) - KT $O(x^2)$ n=201



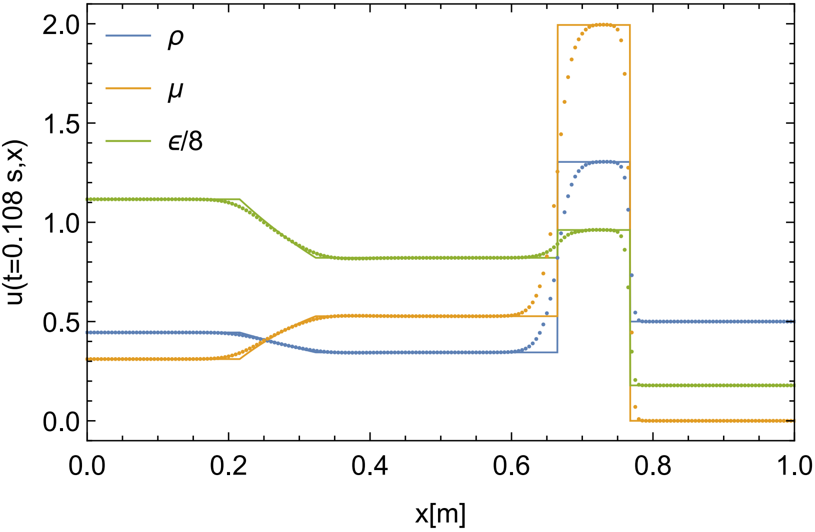
Euler shock tube (EST.Lax) - KT $O(x^2)$ n=201



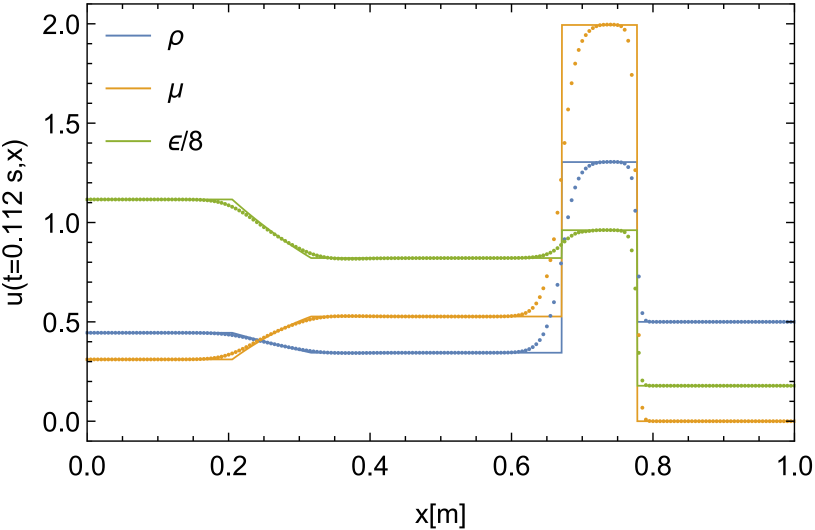
Euler shock tube (EST.Lax) - KT $O(x^2)$ n=201



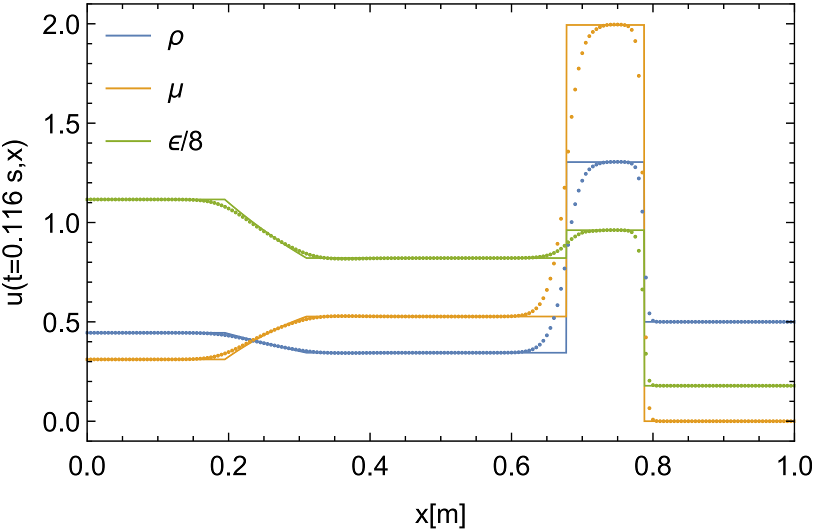
Euler shock tube (EST.Lax) - KT $O(x^2)$ n=201



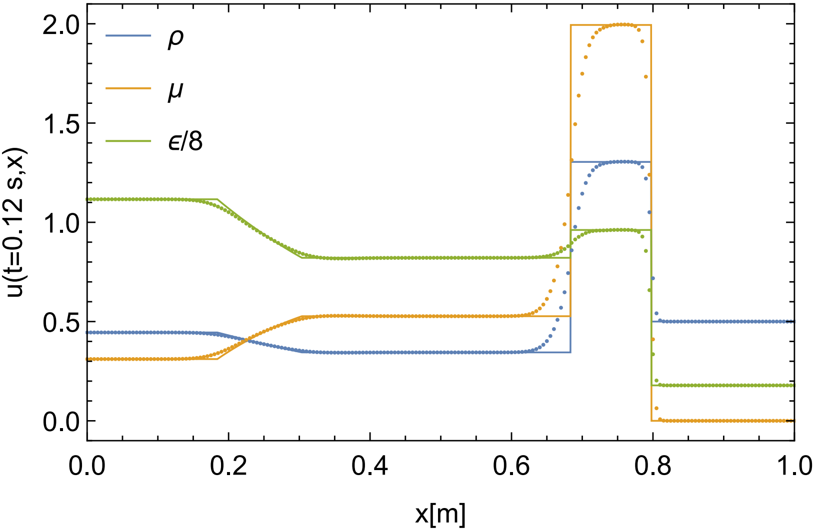
Euler shock tube (EST.Lax) - KT $O(x^2)$ n=201



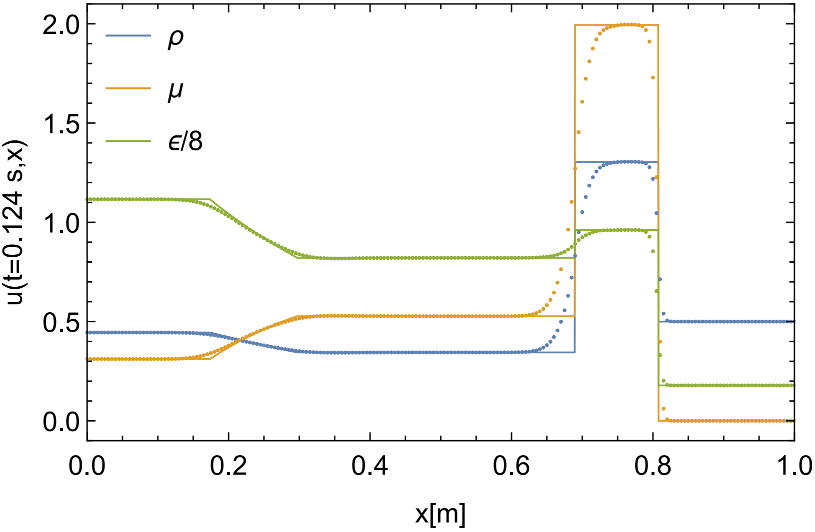
Euler shock tube (EST.Lax) - KT $O(x^2)$ n=201

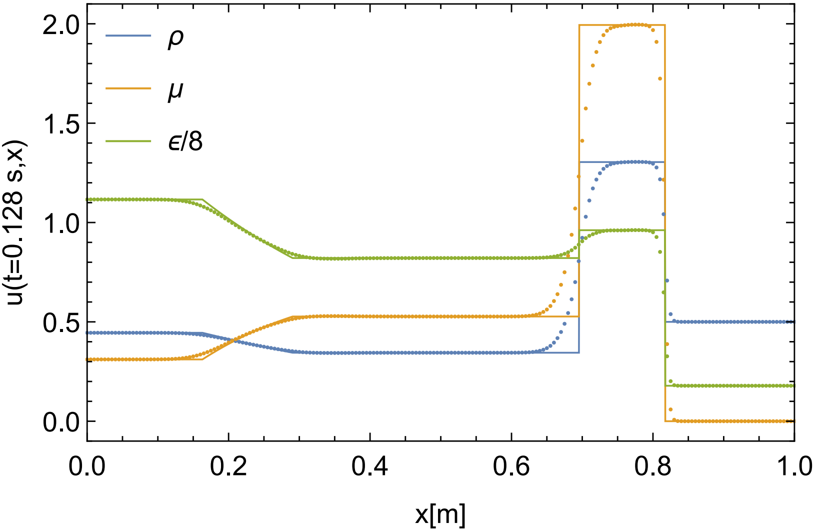


Euler shock tube (EST.Lax) - KT $O(x^2)$ n=201



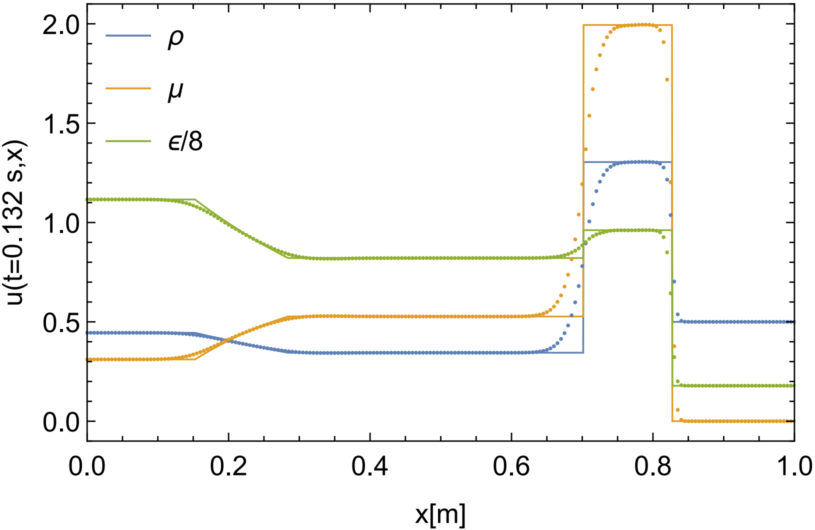
Euler shock tube (EST.Lax) - KT $O(x^2)$ n=201



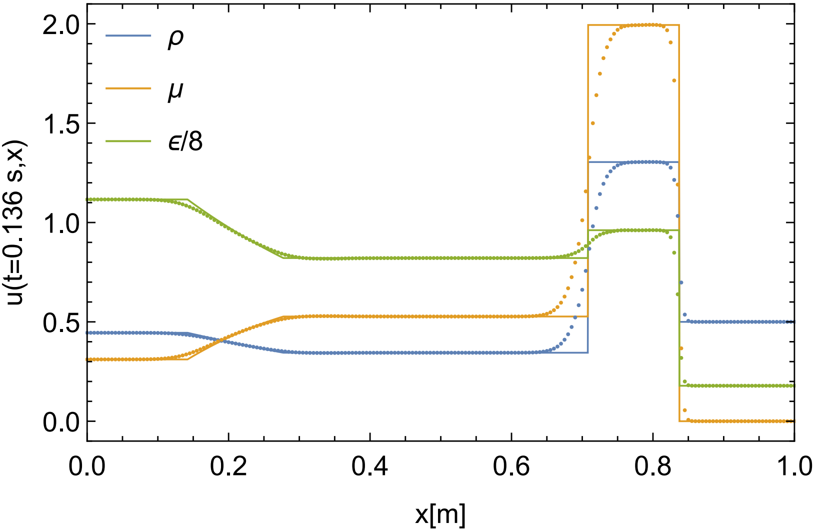


ρ
μ
ϵ/8
u(t=0.128 s,x)
x[m]
0.0
0.5
1.0
1.5
2.0
0.0
0.2
0.4
0.6
0.8
1.0

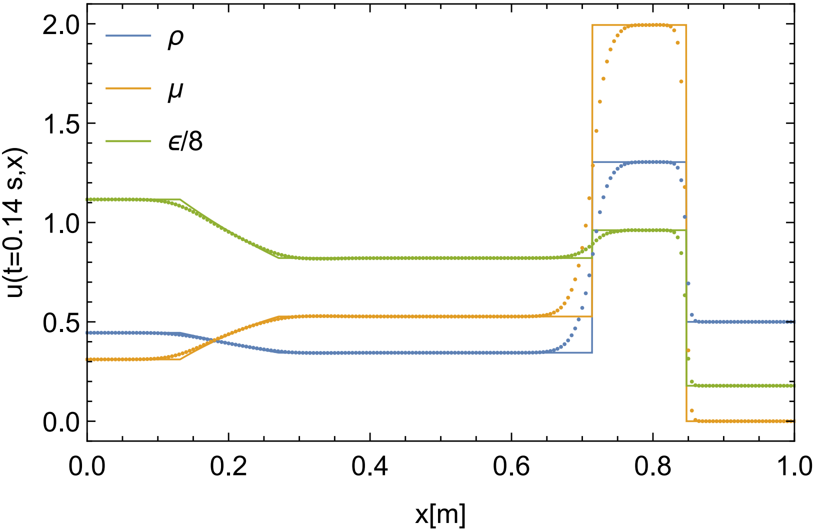
Euler shock tube (EST.Lax) - KT $O(x^2)$ n=201



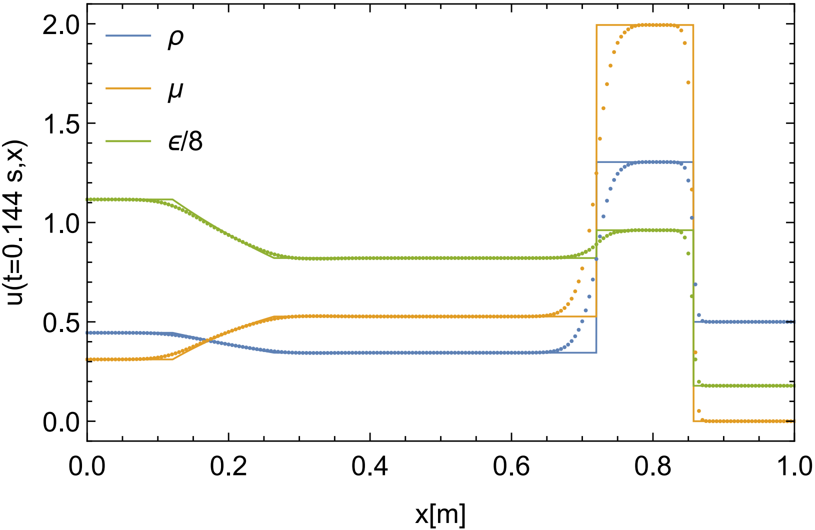
Euler shock tube (EST.Lax) - KT $O(x^2)$ n=201



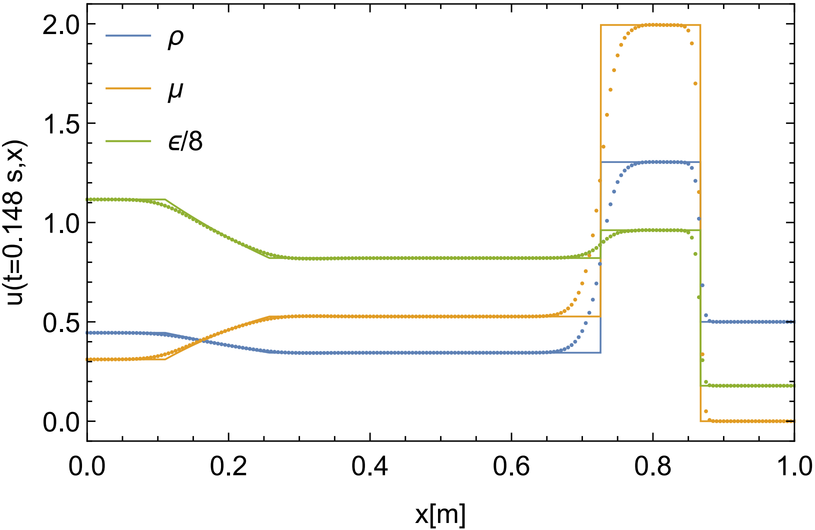
Euler shock tube (EST.Lax) - KT $O(x^2)$ n=201

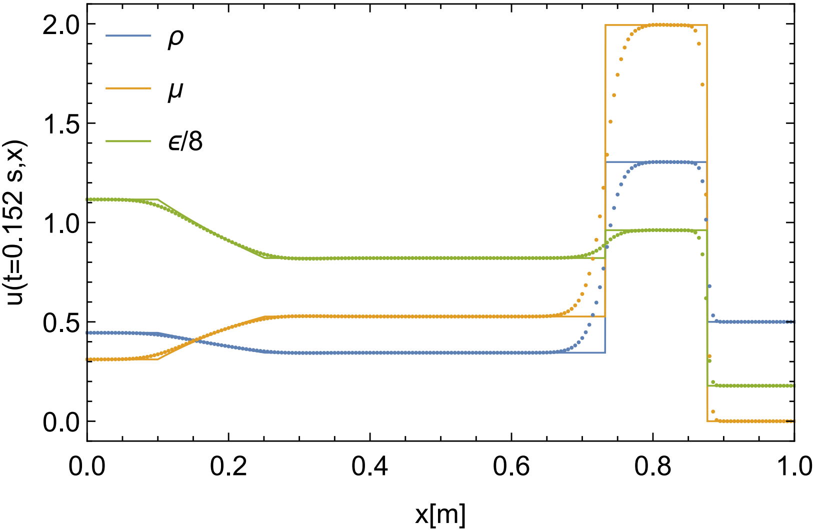


Euler shock tube (EST.Lax) - KT $O(x^2)$ n=201

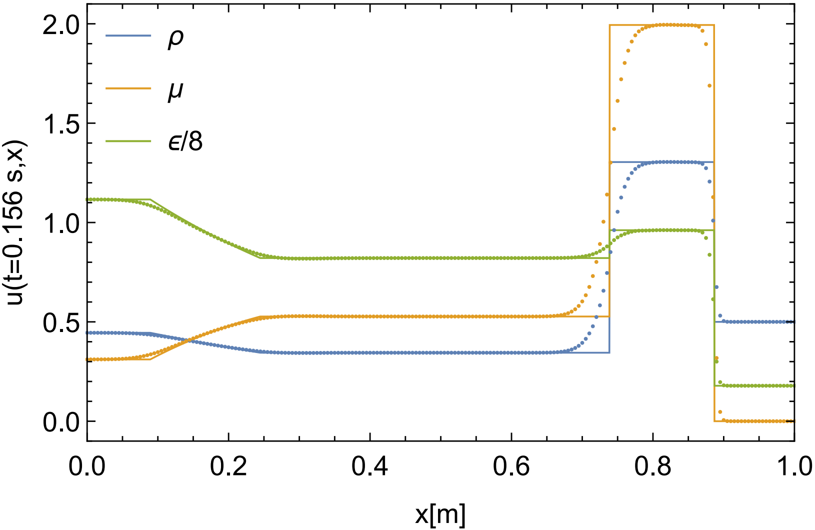


Euler shock tube (EST.Lax) - KT $O(x^2)$ n=201

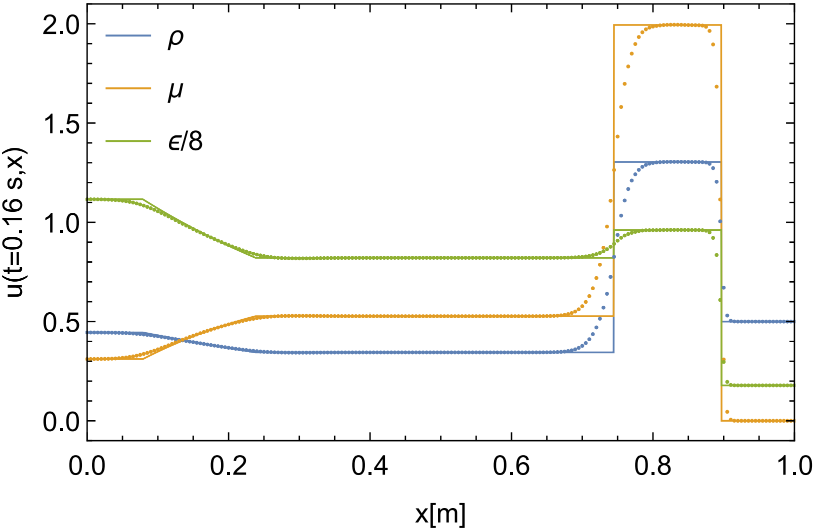



ρ
μ
ϵ/8
u(t=0.152 s,x)
x[m]
0.0
0.5
1.0
1.5
2.0
0.0
0.2
0.4
0.6
0.8
1.0

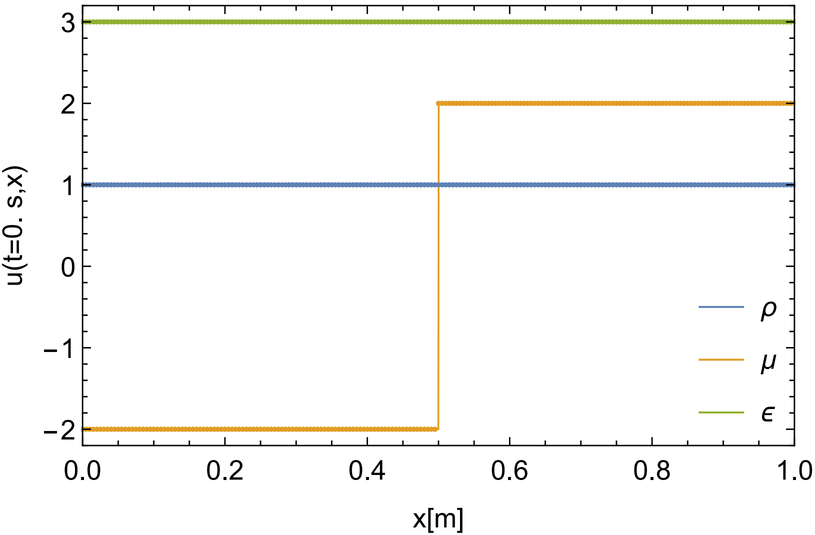
Euler shock tube (EST.Lax) - KT $O(x^2)$ n=201



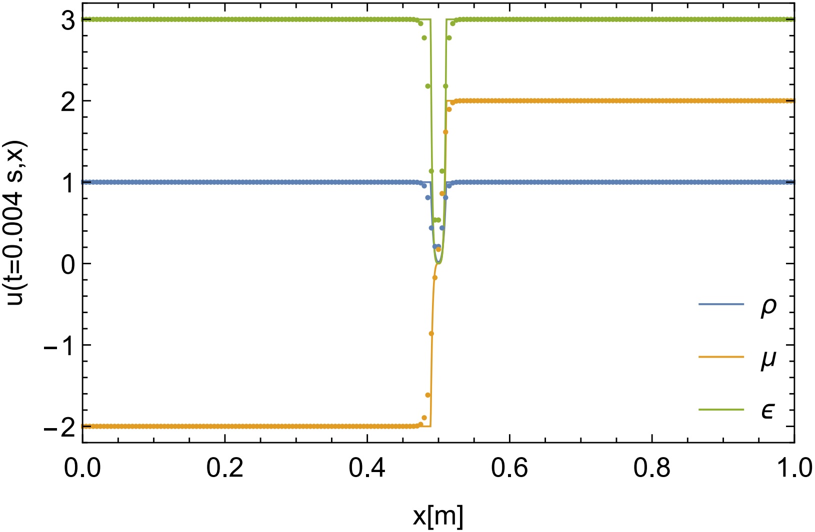
Euler shock tube (EST.Lax) - KT $O(x^2)$ n=201



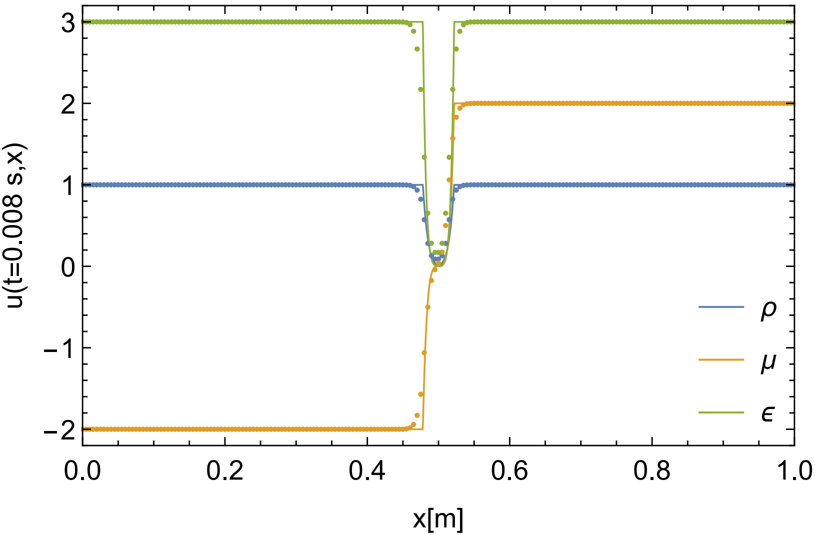
Euler shock tube (EST.Toro2) - KT O(2) n=201



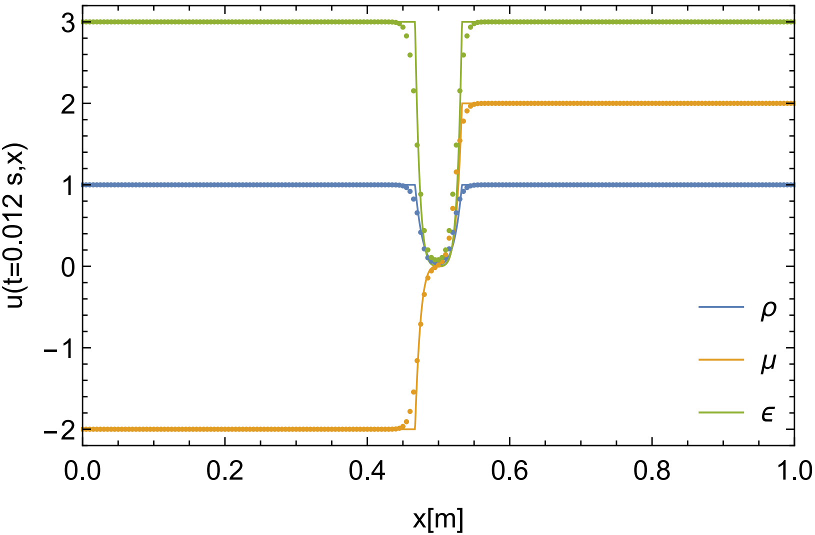
Euler shock tube (EST.Toro2) - KT O(2) n=201



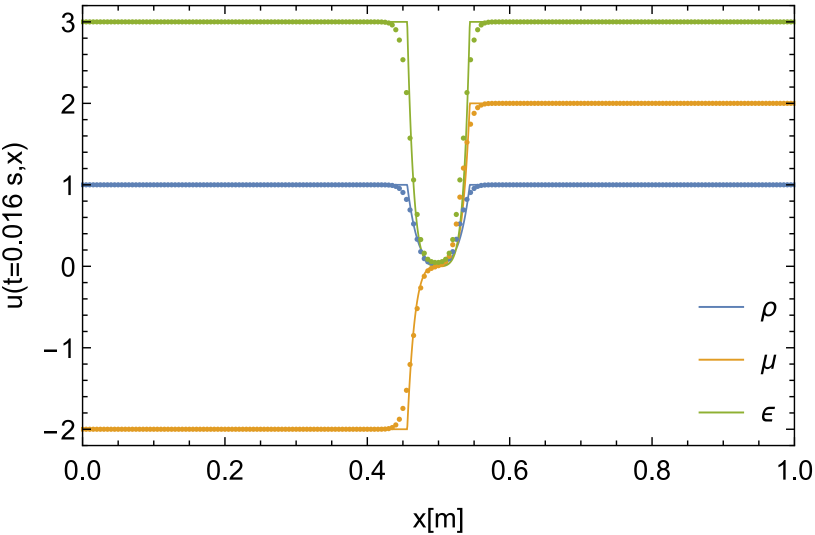
Euler shock tube (EST.Toro2) - KT O(2) n=201



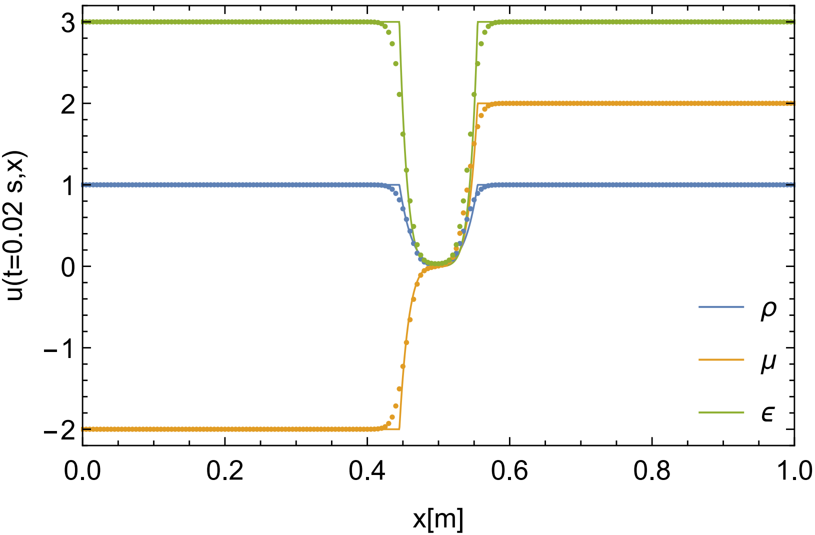
Euler shock tube (EST.Toro2) - KT O(2) n=201



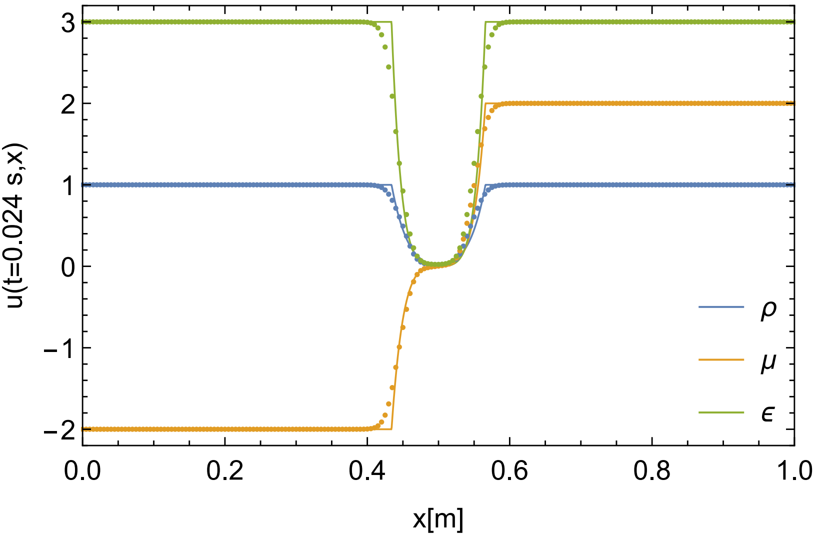
Euler shock tube (EST.Toro2) - KT O(2) n=201

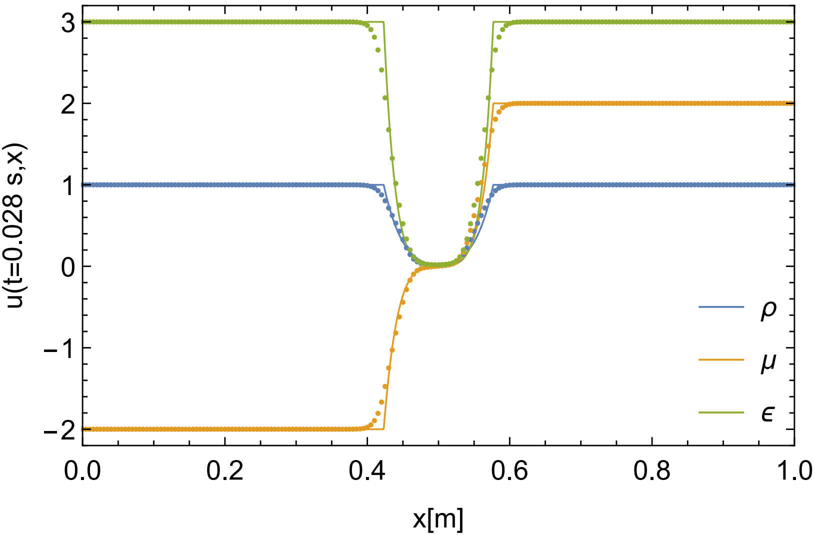


Euler shock tube (EST.Toro2) - KT O(2) n=201



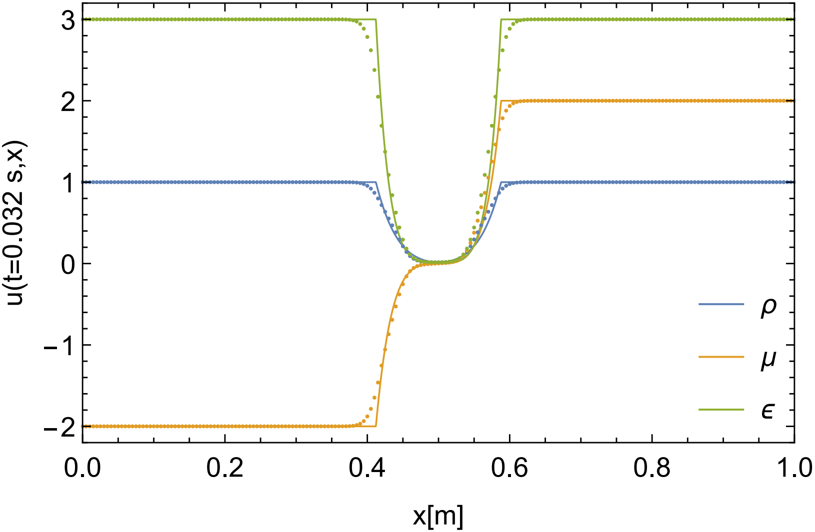
Euler shock tube (EST.Toro2) - KT O(2) n=201



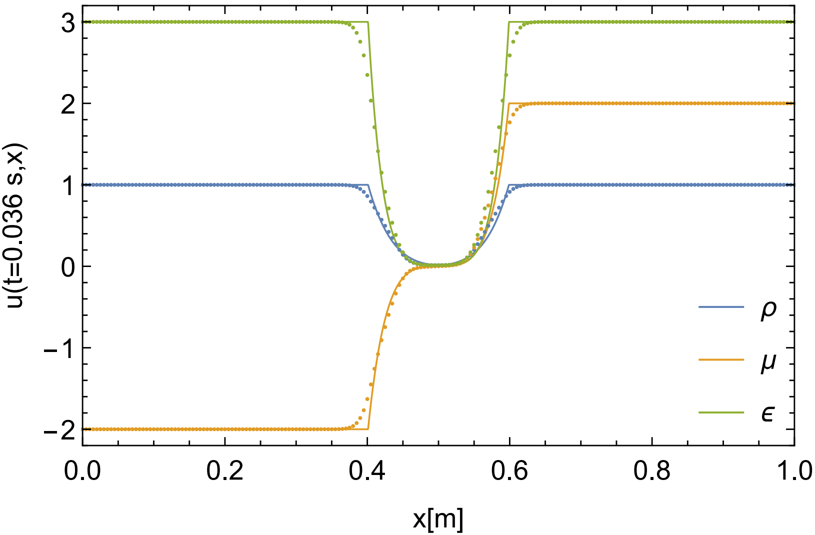


u(t=0.028 s,x)
x[m]
0.0
0.2
0.4
0.6
0.8
1.0
−2
−1
0
1
2
3
ρ
μ
ϵ

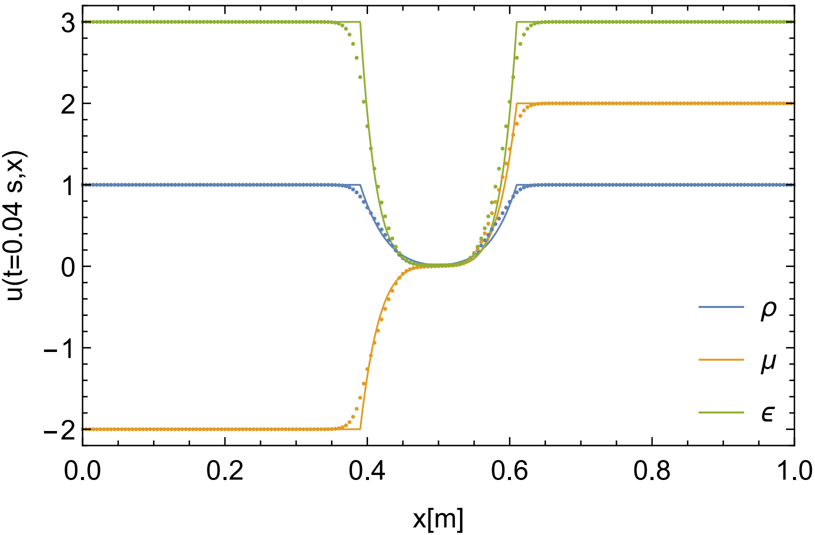
Euler shock tube (EST.Toro2) - KT O(2) n=201



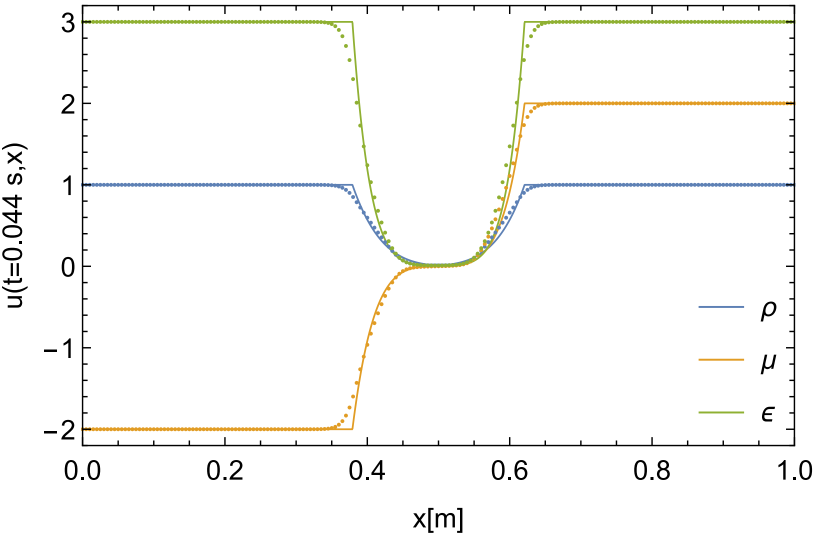
Euler shock tube (EST.Toro2) - KT O(2) n=201



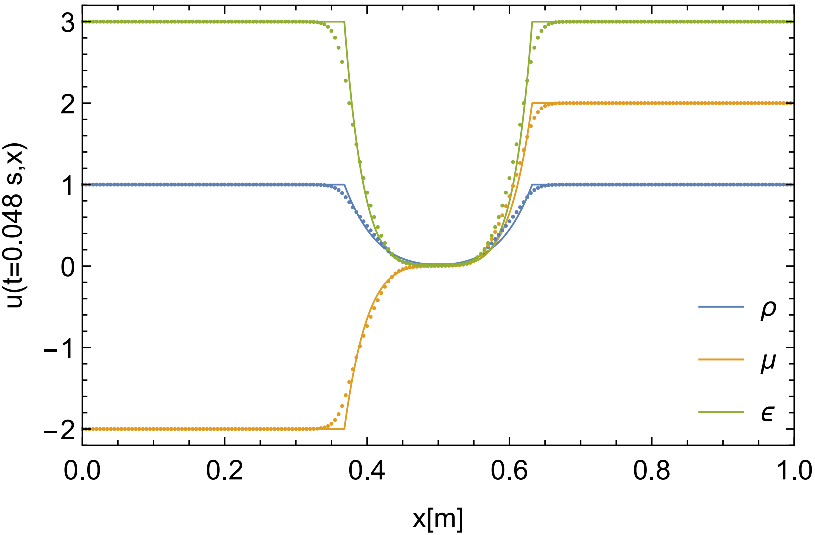
Euler shock tube (EST.Toro2) - KT O(2) n=201



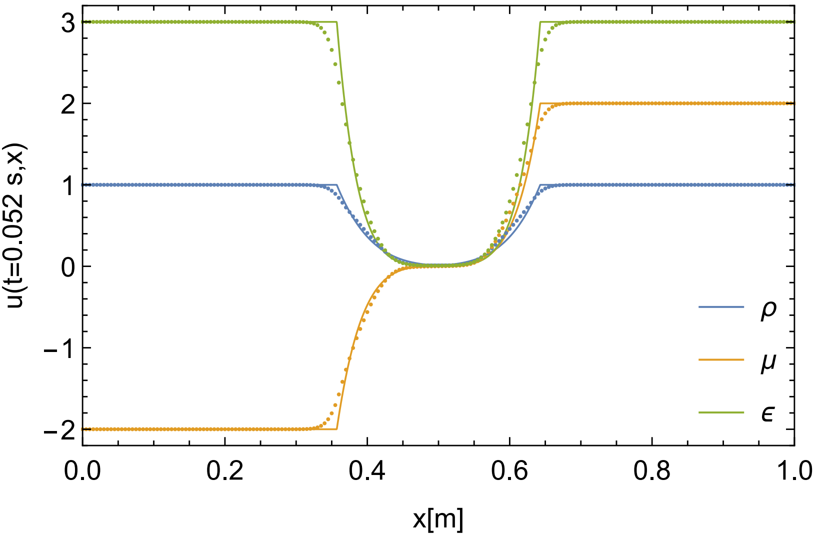
Euler shock tube (EST.Toro2) - KT O(2) n=201



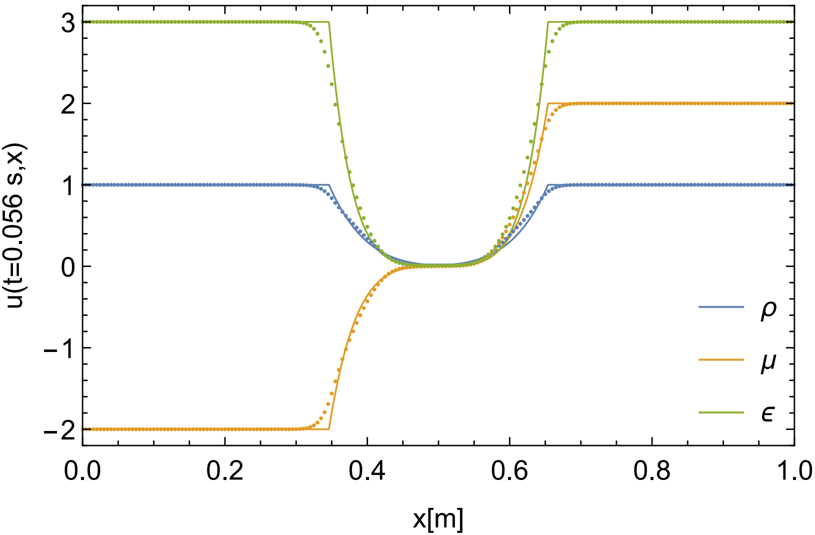
Euler shock tube (EST.Toro2) - KT O(2) n=201



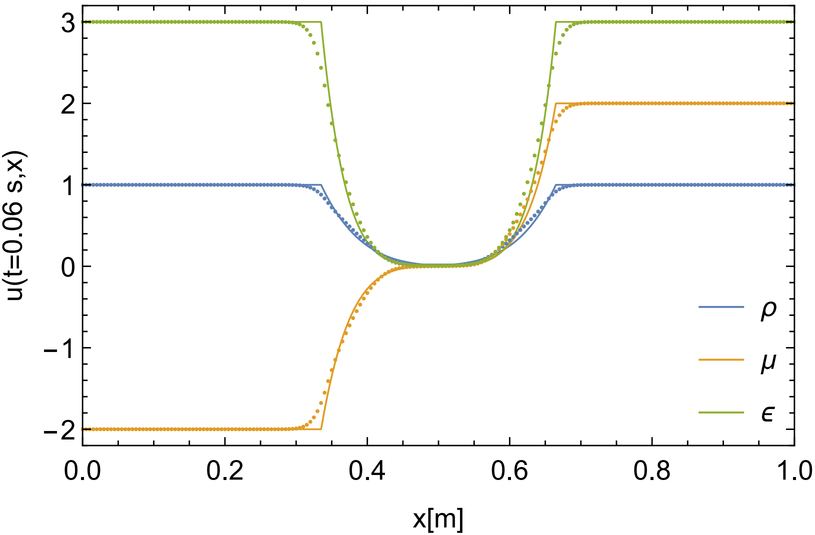
Euler shock tube (EST.Toro2) - KT O(2) n=201



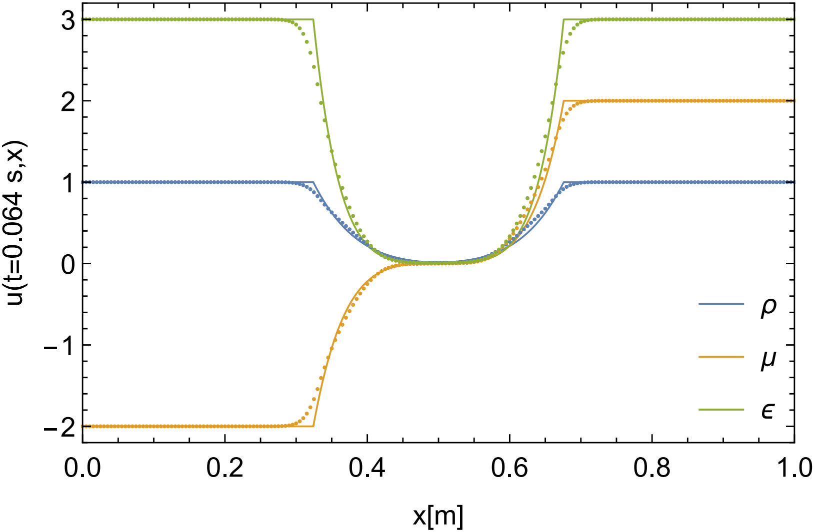
Euler shock tube (EST.Toro2) - KT O(2) n=201



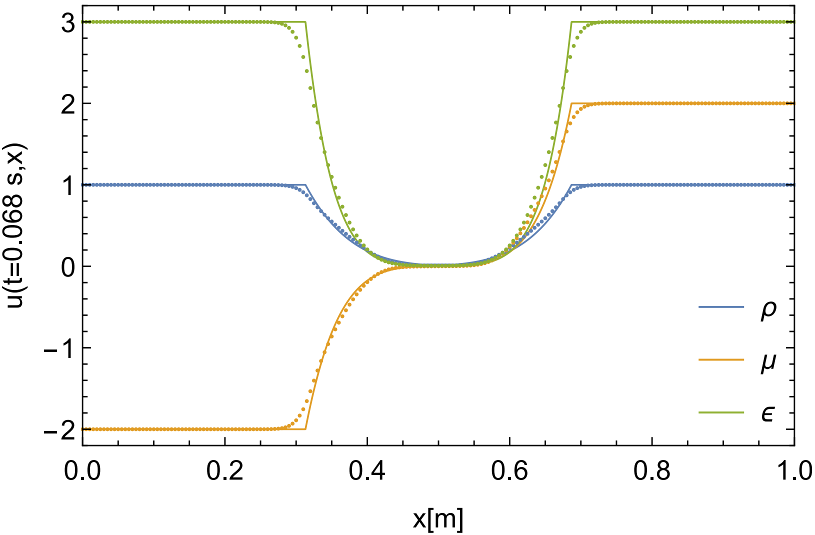
Euler shock tube (EST.Toro2) - KT O(2) n=201



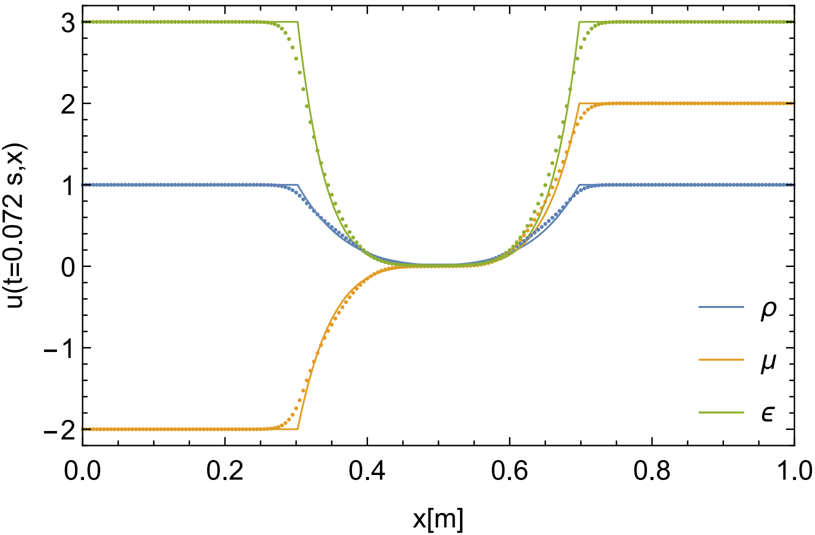
Euler shock tube (EST.Toro2) - KT O(2) n=201



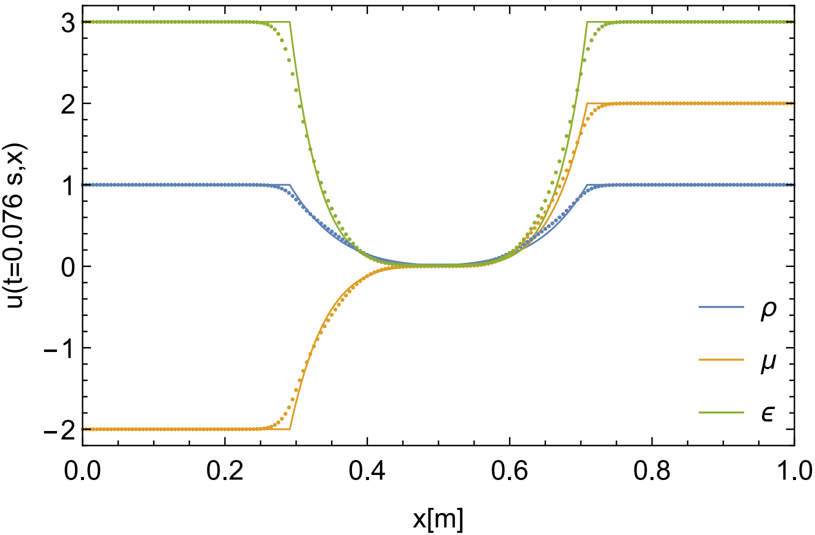
Euler shock tube (EST.Toro2) - KT O(2) n=201



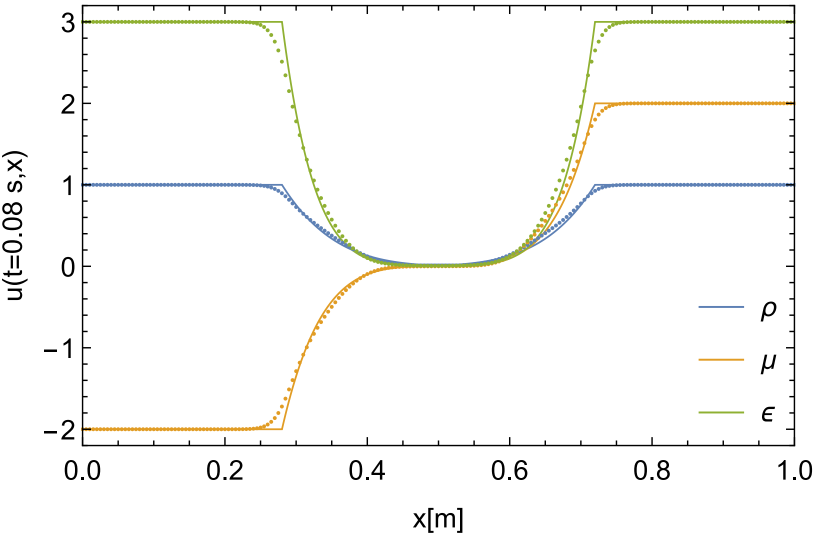
Euler shock tube (EST.Toro2) - KT O(2) n=201



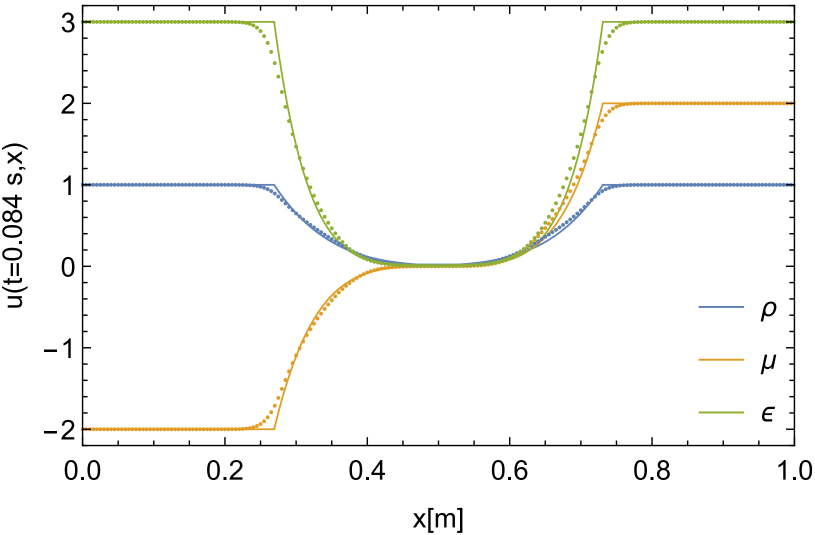
Euler shock tube (EST.Toro2) - KT O(2) n=201



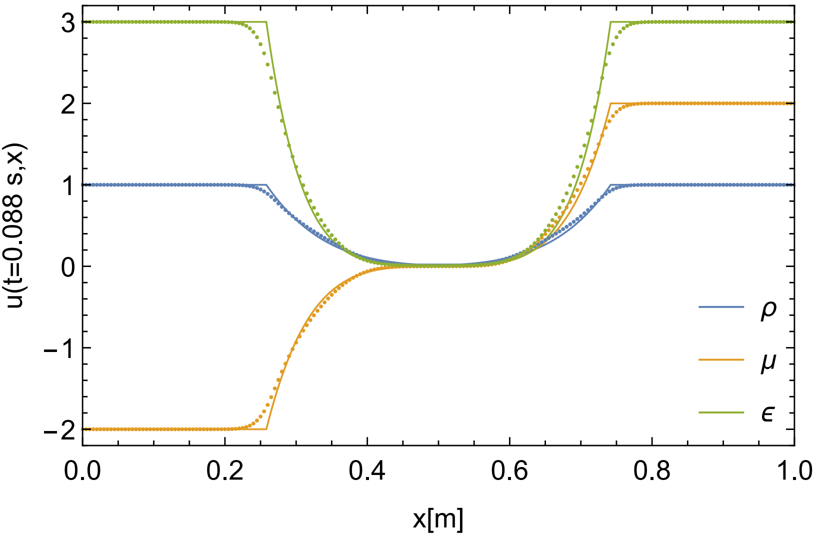
Euler shock tube (EST.Toro2) - KT O(2) n=201



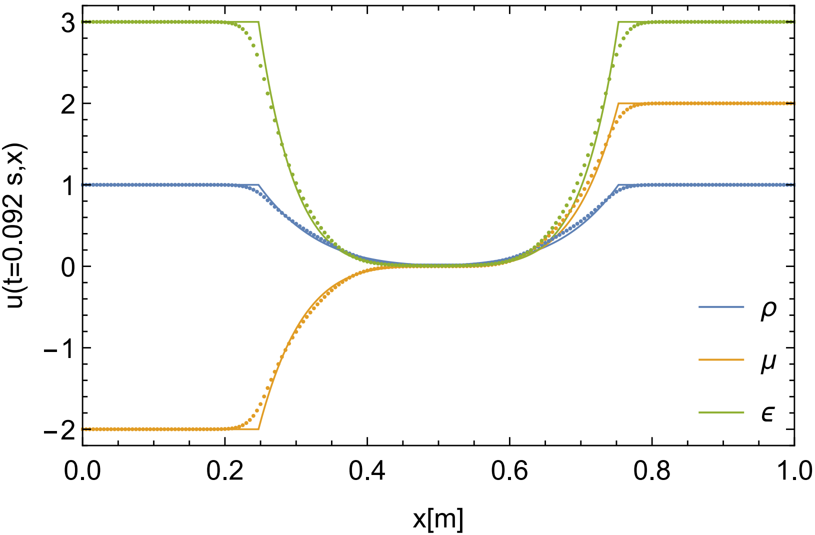
Euler shock tube (EST.Toro2) - KT O(2) n=201



Euler shock tube (EST.Toro2) - KT O(2) n=201



Euler shock tube (EST.Toro2) - KT O(2) n=201



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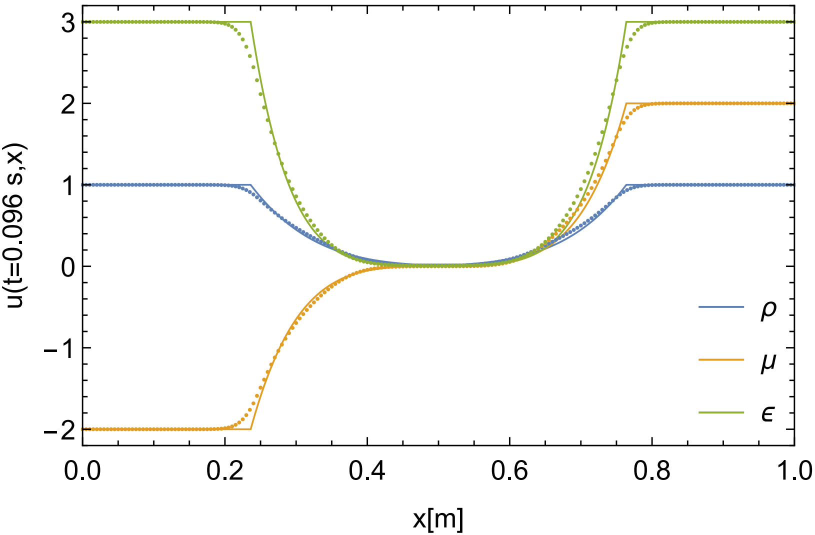

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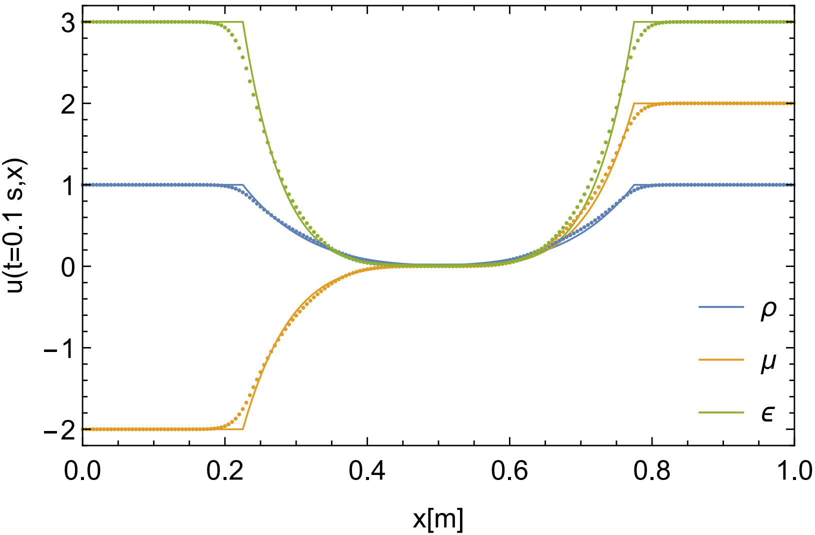
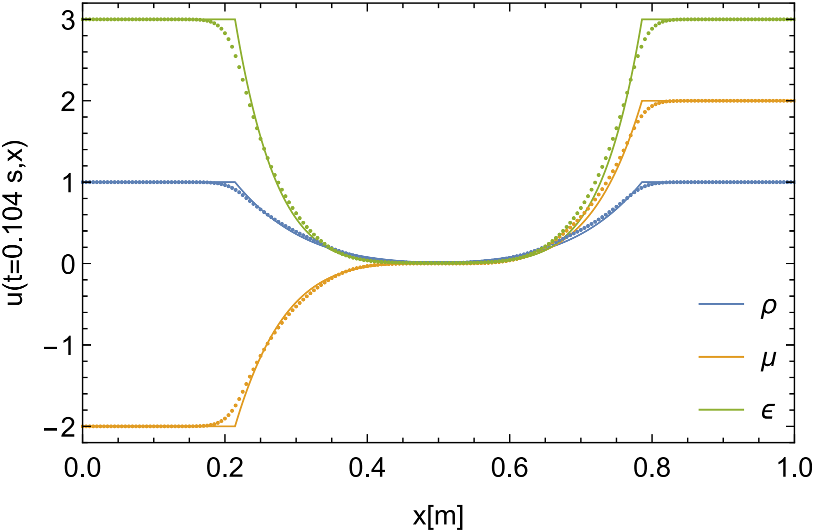
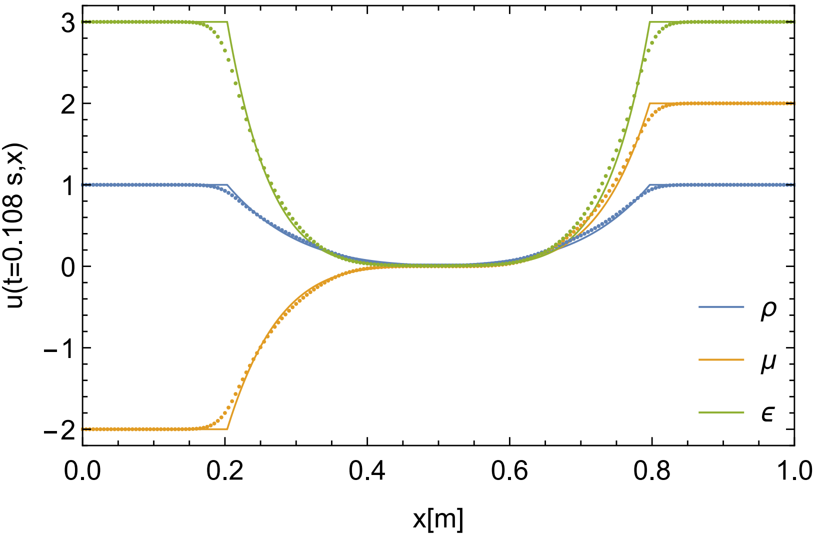


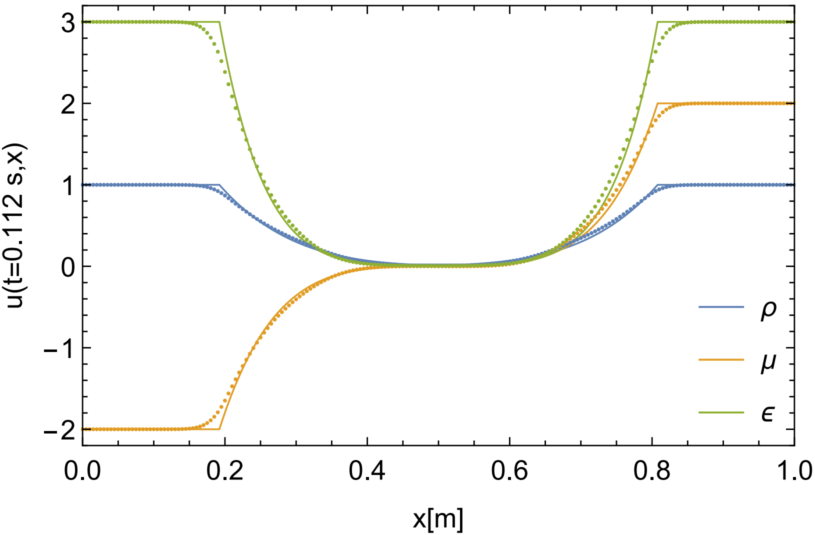
Euler shock tube (EST.Toro2) - KT O(2) n=201



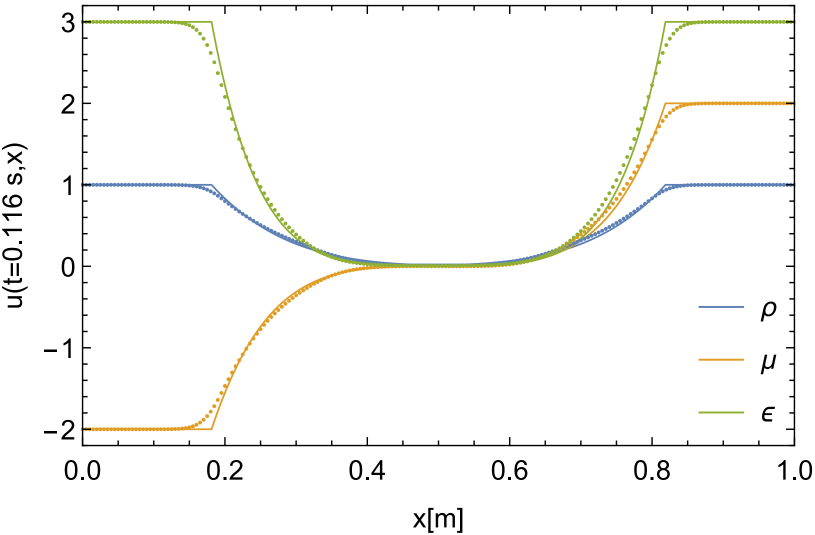
Euler shock tube (EST.Toro2) - KT O(2) n=201



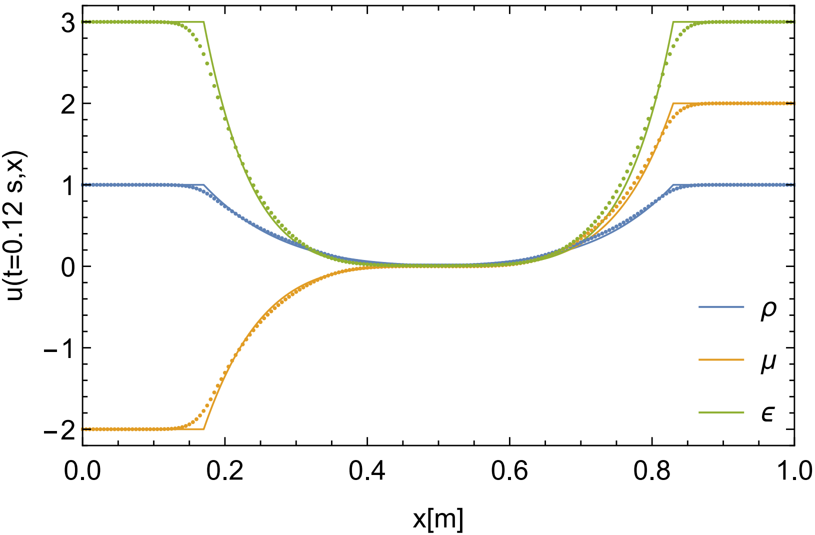
Euler shock tube (EST.Toro2) - KT O(2) n=201



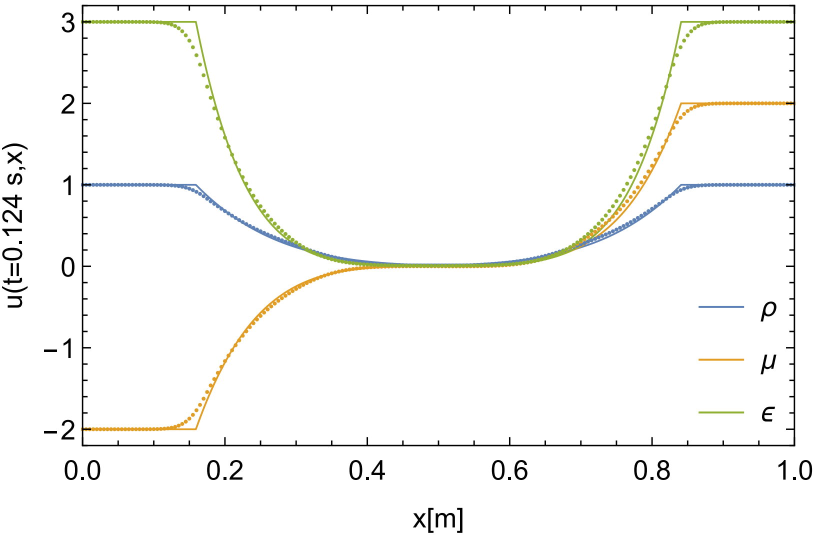
Euler shock tube (EST.Toro2) - KT O(2) n=201



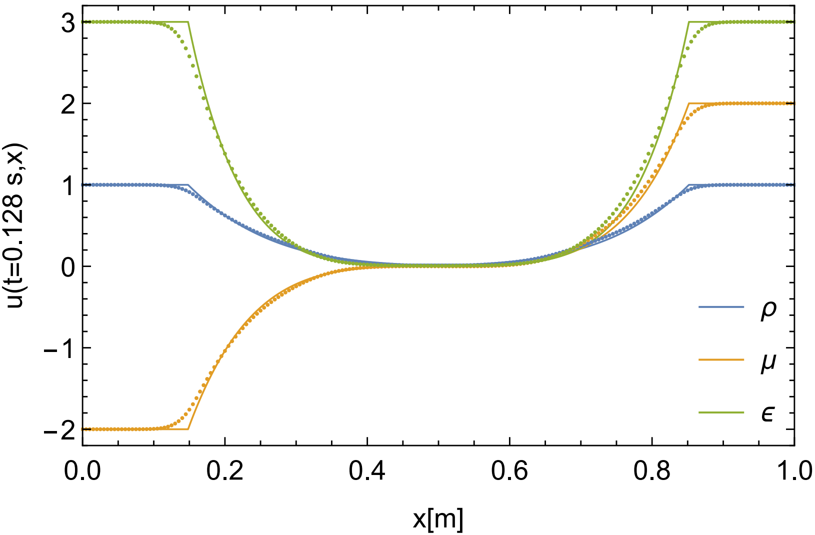
Euler shock tube (EST.Toro2) - KT O(2) n=201



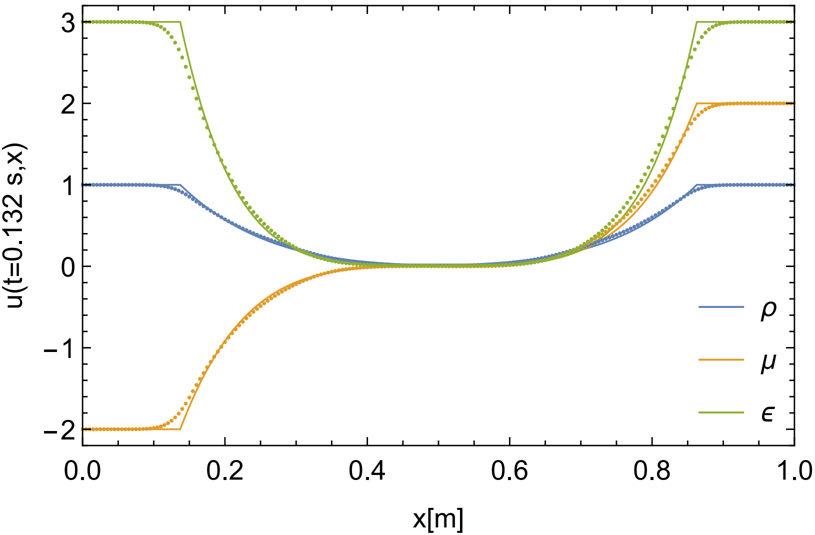
Euler shock tube (EST.Toro2) - KT O(2) n=201



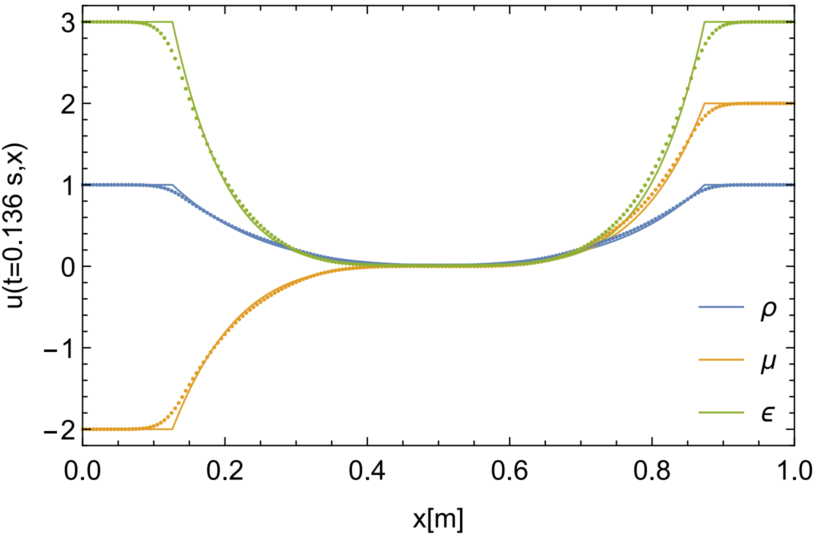
Euler shock tube (EST.Toro2) - KT O(2) n=201



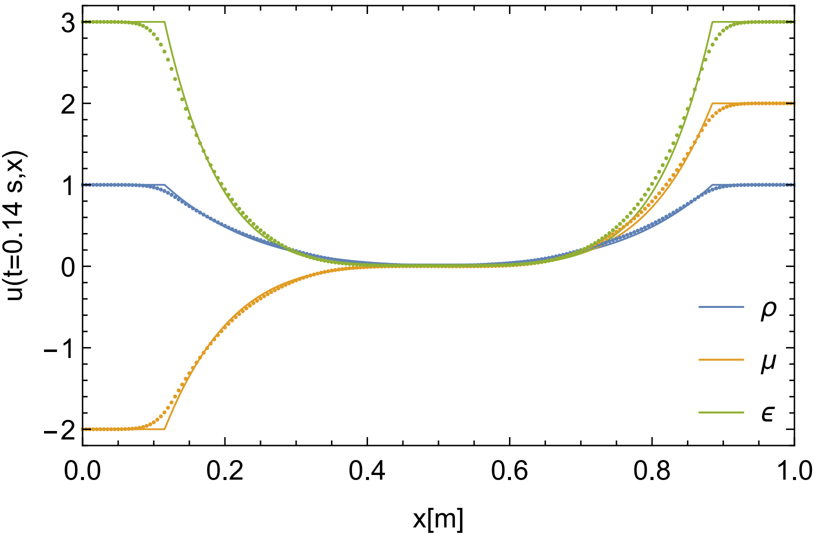
Euler shock tube (EST.Toro2) - KT O(2) n=201



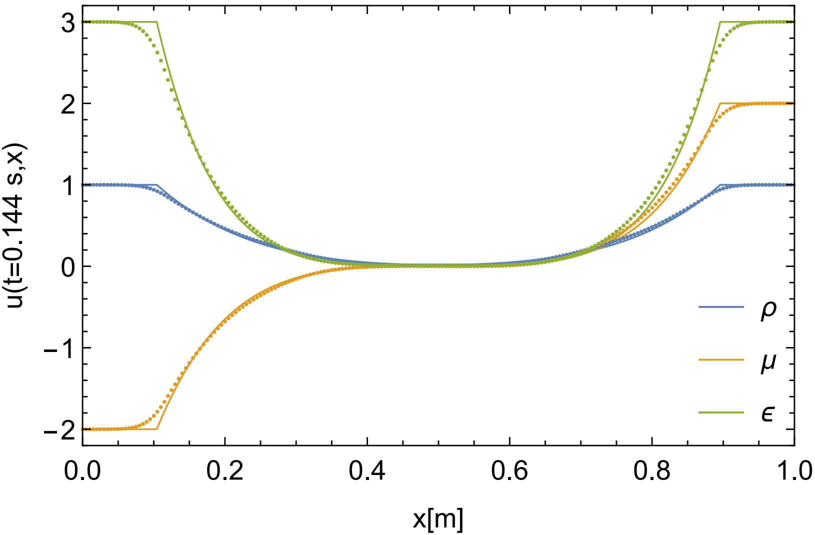
Euler shock tube (EST.Toro2) - KT O(2) n=201

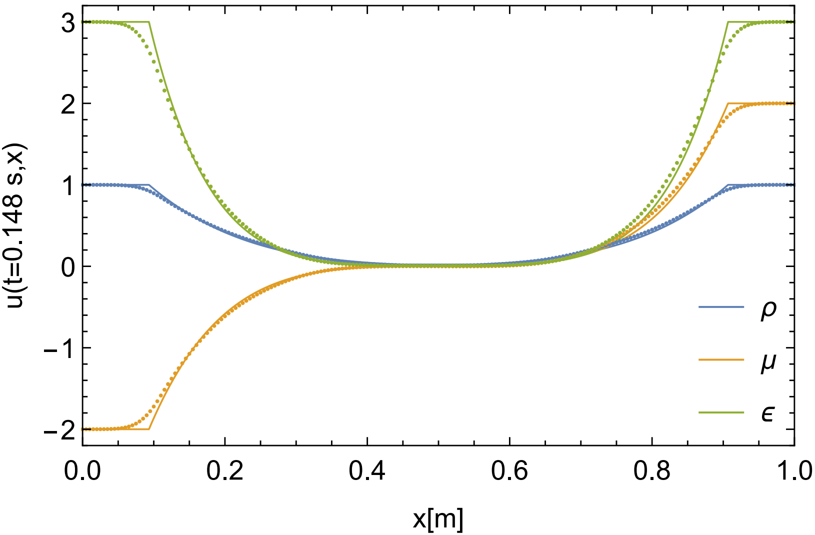


Euler shock tube (EST.Toro2) - KT O(2) n=201



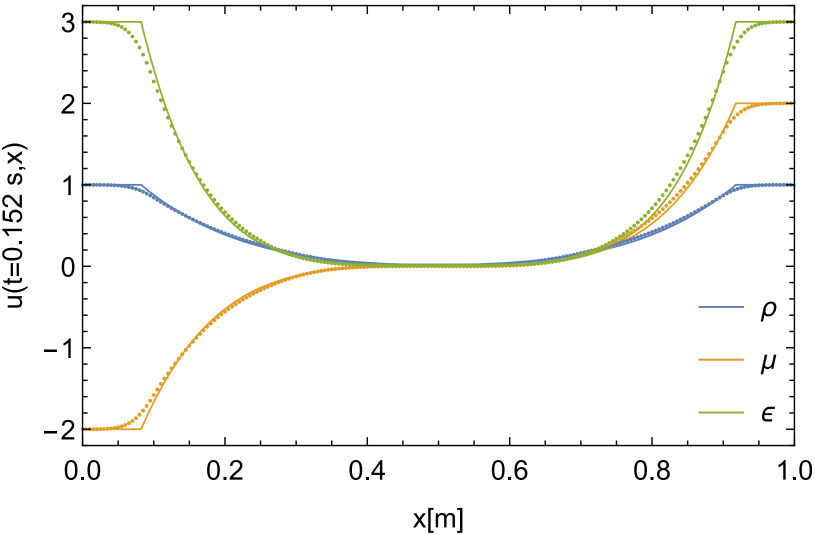
Euler shock tube (EST.Toro2) - KT O(2) n=201



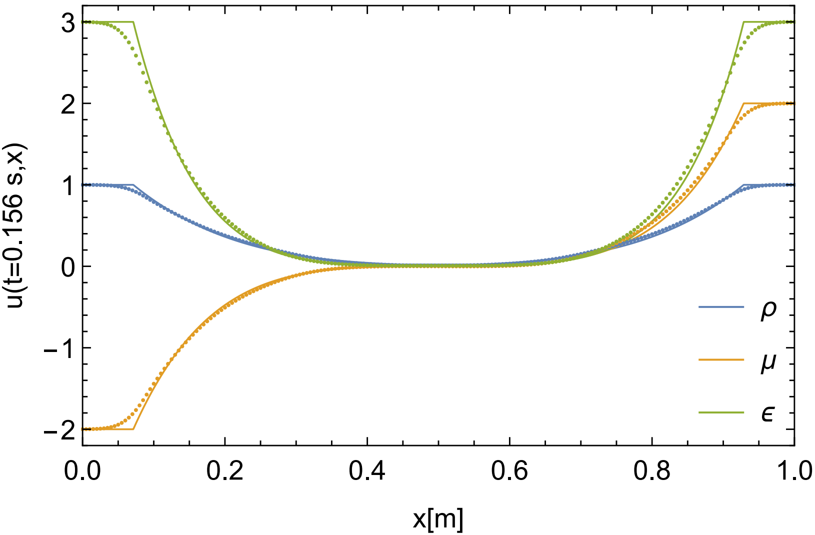


u(t=0.148 s,x)
x[m]
3
2
1
0
−1
−2
0.0
0.2
0.4
0.6
0.8
1.0
ρ
μ
ϵ

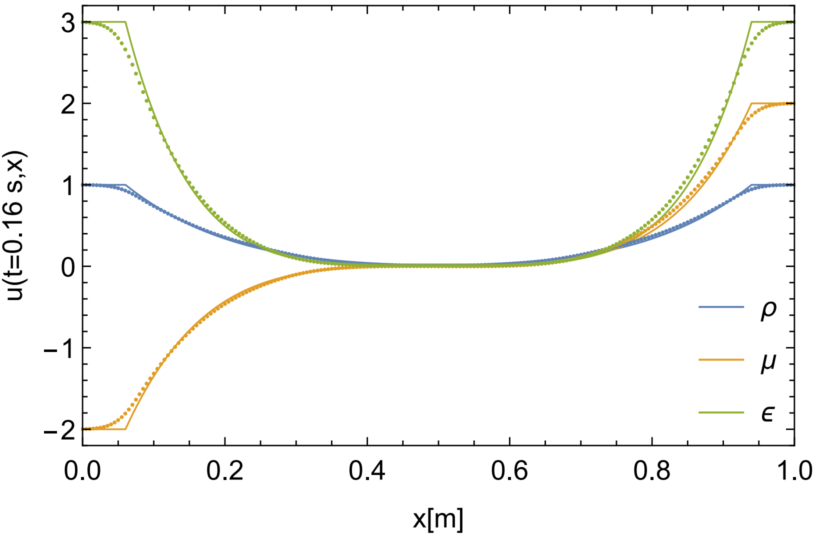
Euler shock tube (EST.Toro2) - KT O(2) n=201



Euler shock tube (EST.Toro2) - KT O(2) n=201



Euler shock tube (EST.Toro2) - KT O(2) n=201



O(N) model - LPA flow equation $d > 0$

$$\partial_t u(t,x) + c_d \partial_x \frac{\mathrm{e}^{-dt}}{1 + \mathrm{e}^{2t} u(t,x)} = -\frac{c_d}{N-1} \partial_x \frac{\mathrm{e}^{-dt}}{1 + \mathrm{e}^{2t}(u(t,x) + 2x\partial_x u(t,x))} \tag{12}$$

- LPA flow equation of the O(N) model in $d > 0$ dimensions using d-dim. Litim regulator with

$$t \equiv -\log k/\Lambda \tag{13}$$

$$x \equiv \frac{\rho}{(N-1)\Lambda^{d-2}} \tag{14}$$

$$u(t,x) \equiv \frac{\partial U(t,\rho)}{\Lambda^2 \partial \rho} \tag{15}$$

$$c_d \equiv \frac{2^{-d}\pi^{-d/2}}{\Gamma(1+d/2)} \tag{16}$$

$O(\infty)$ model - $d = 3$ initial conditions

- Riemann problem (restored phase for $O(\infty)$, d=3)

$$u(t=0,x) = \begin{cases} 0.0 & 0.02 < x < 0.05 \\ 0.1 & \text{otherwise} \end{cases} \tag{GW.2a}$$

- Quartic potential (restored phase for $O(\infty)$, d=3)

$$u(t=0,x) = 1.0x - 0.1 \tag{GW.3a}$$

- Sextic potential (restored phase for $O(\infty)$, d=3)

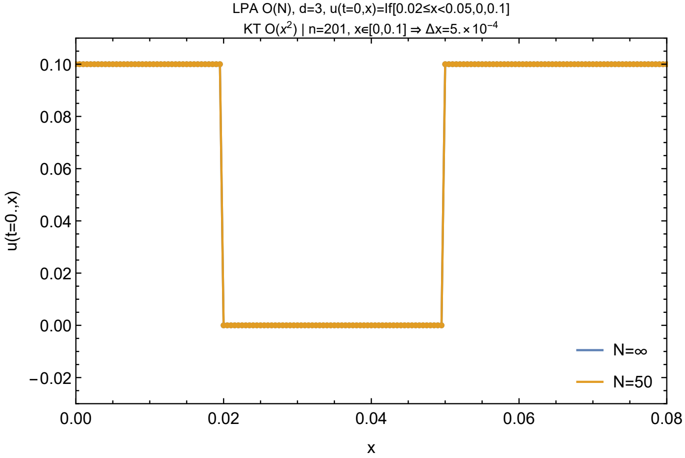
$$u(t=0,x) = 1.0x^2 - 0.103x + 0.0024 \tag{GW.8c}$$

- Sextic potential (broken phase for $O(\infty)$, d=3)

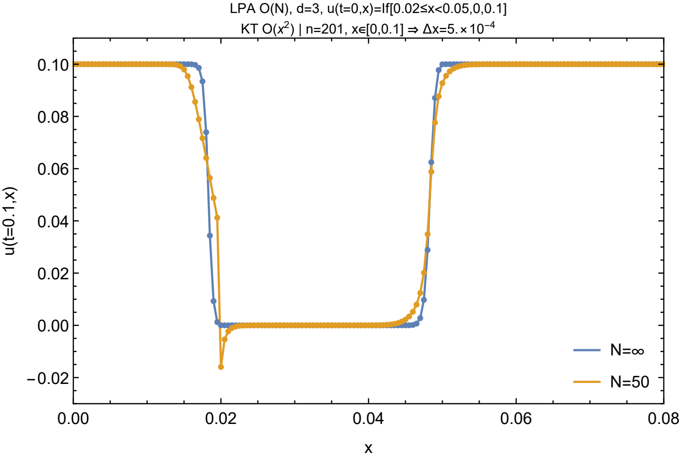
$$u(t=0,x) = 1.0x^2 - 0.105x + 0.0024 \tag{GW.8d}$$

ICs for various figures in E. Grossi and N. Wink (2019), arXiv: 1903.09503 [hep-th]

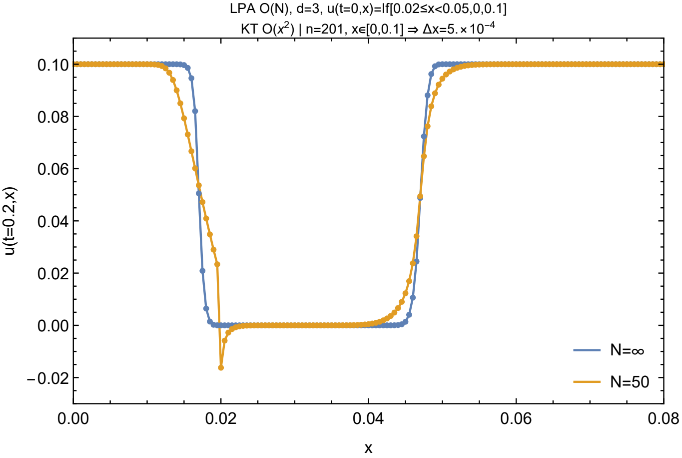
O(N), d=3, (GW.2a) ICs



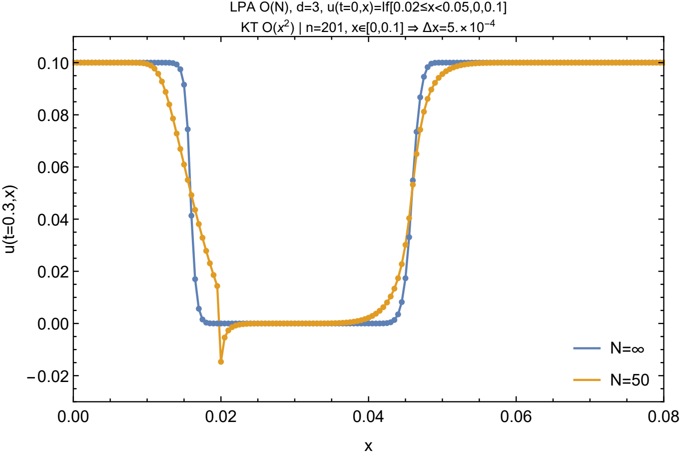
O(N), d=3, (GW.2a) ICs



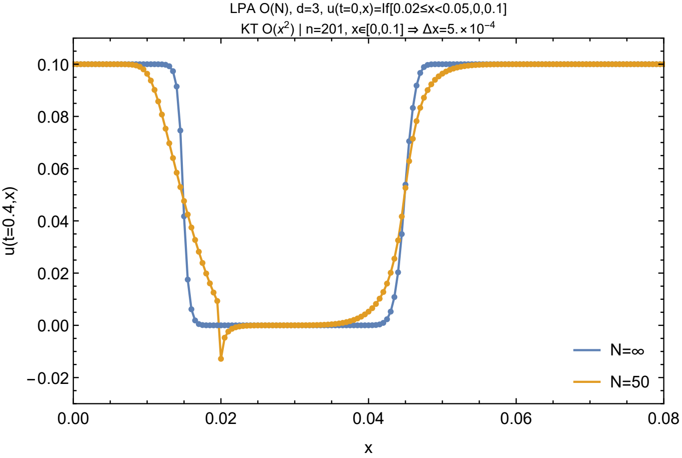
O(N), d=3, (GW.2a) ICs



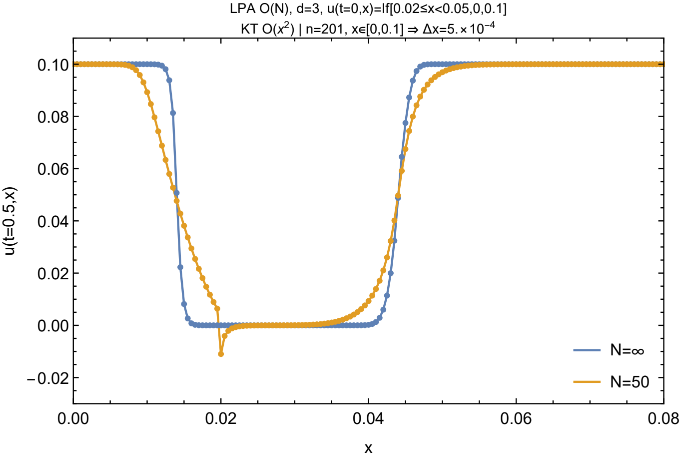
O(N), d=3, (GW.2a) ICs



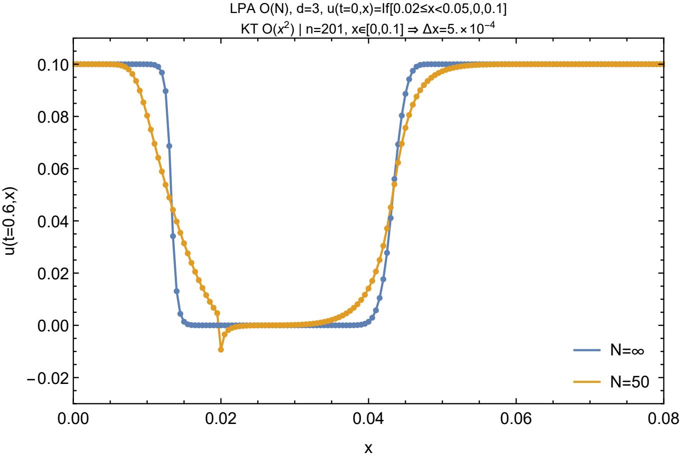
O(N), d=3, (GW.2a) ICs



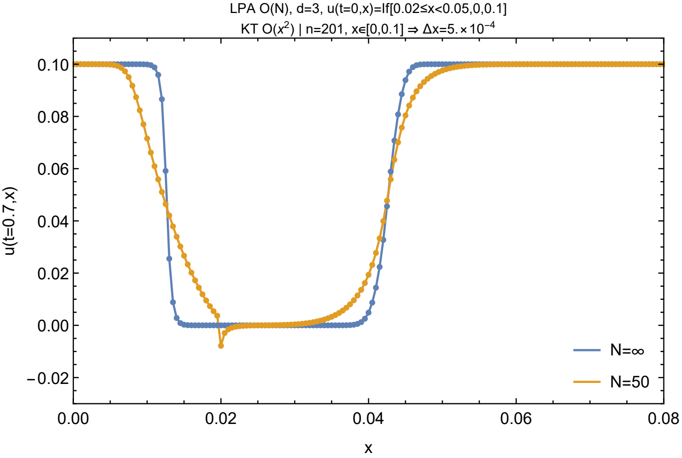
O(N), d=3, (GW.2a) ICs



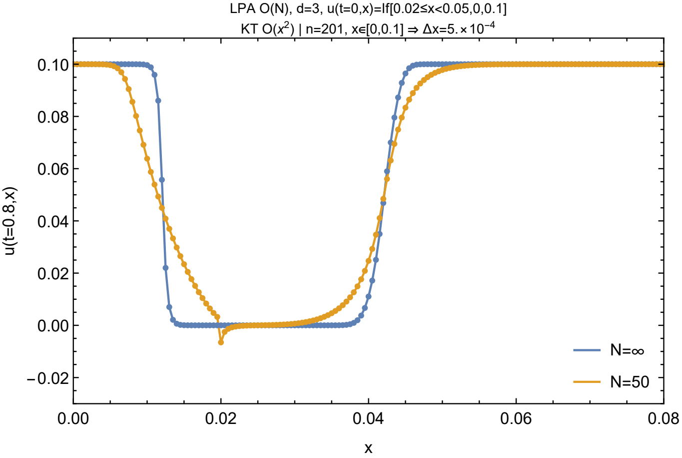
O(N), d=3, (GW.2a) ICs



O(N), d=3, (GW.2a) ICs



O(N), d=3, (GW.2a) ICs



O(N), d=3, (GW.2a) ICs

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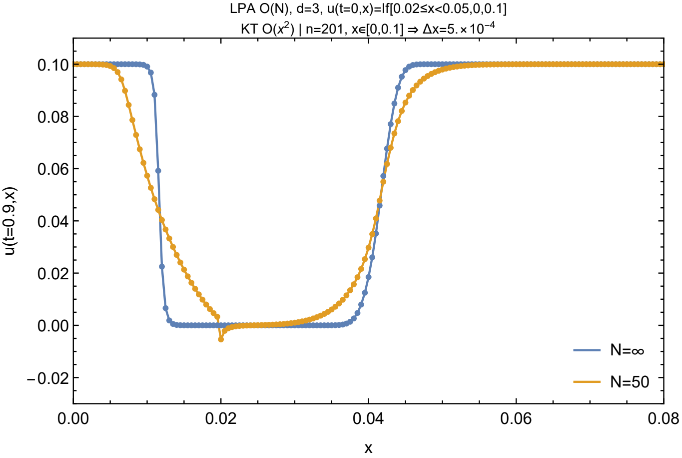
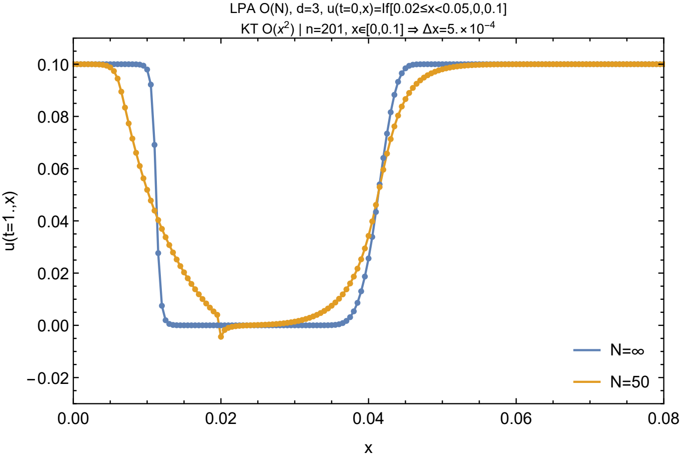
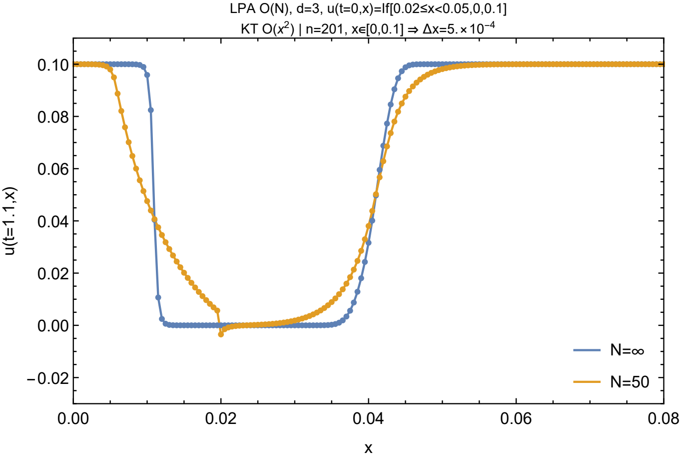


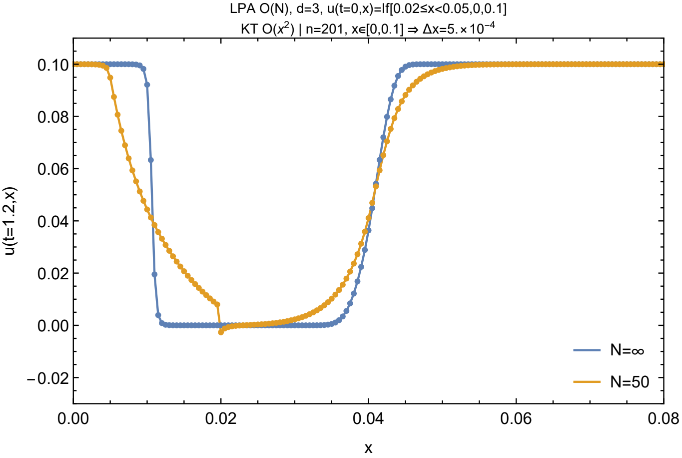
O(N), d=3, (GW.2a) ICs



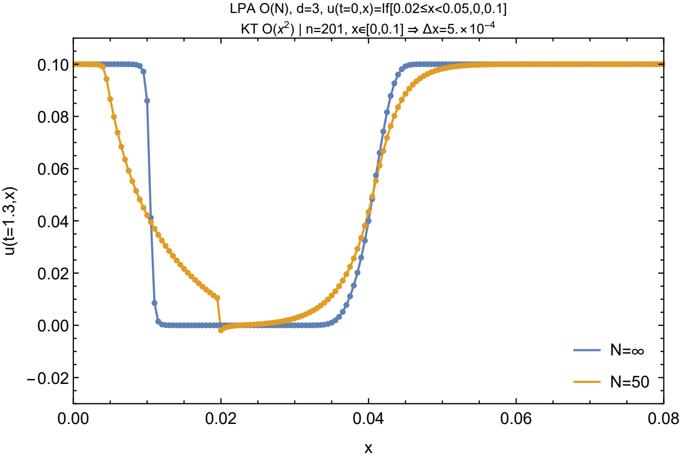
O(N), d=3, (GW.2a) ICs



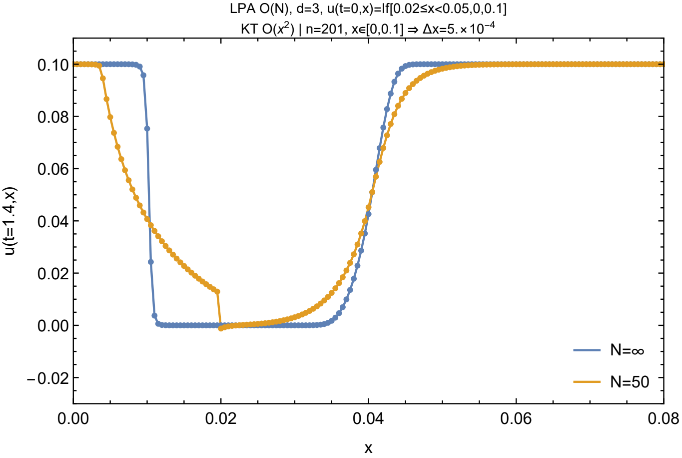
O(N), d=3, (GW.2a) ICs



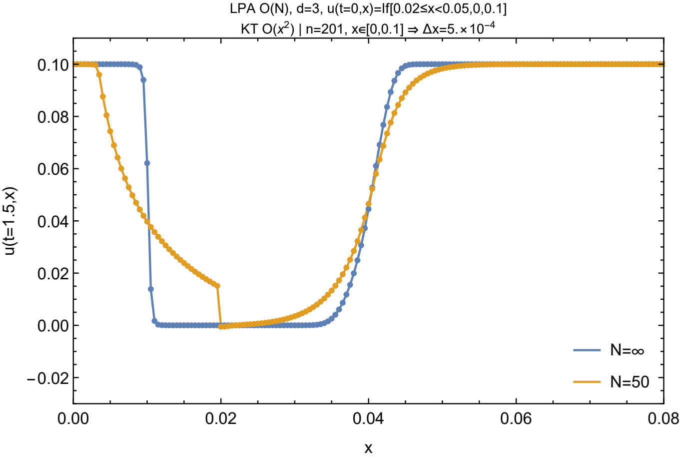
O(N), d=3, (GW.2a) ICs



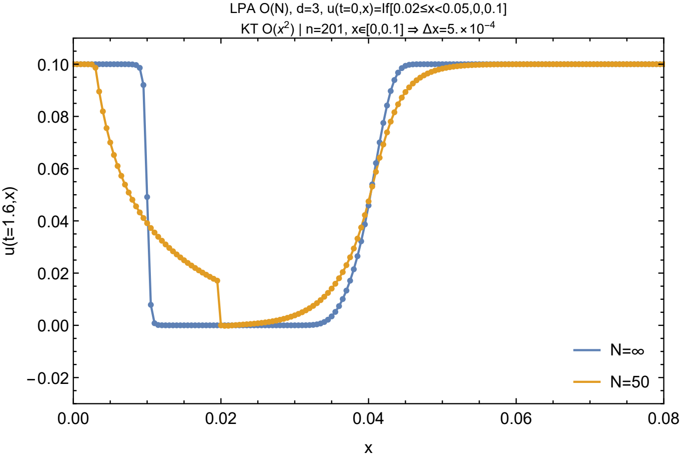
O(N), d=3, (GW.2a) ICs



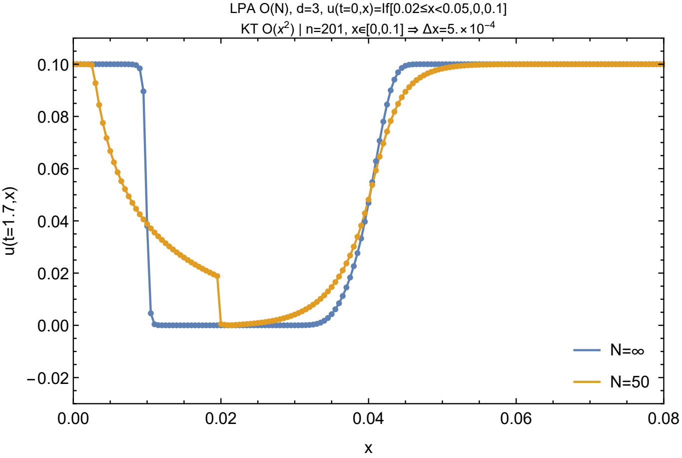
O(N), d=3, (GW.2a) ICs



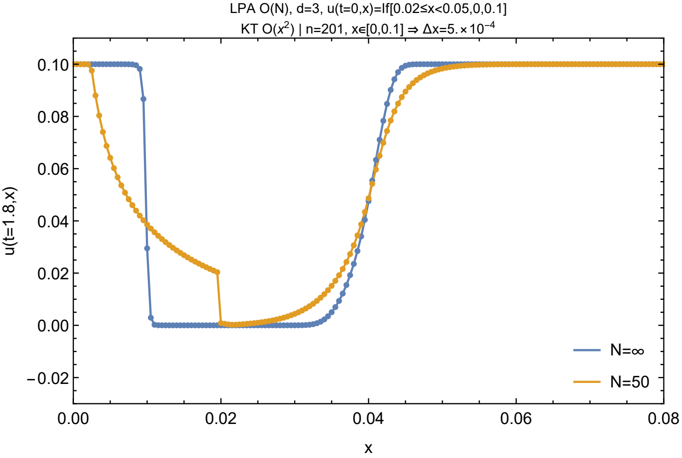
O(N), d=3, (GW.2a) ICs



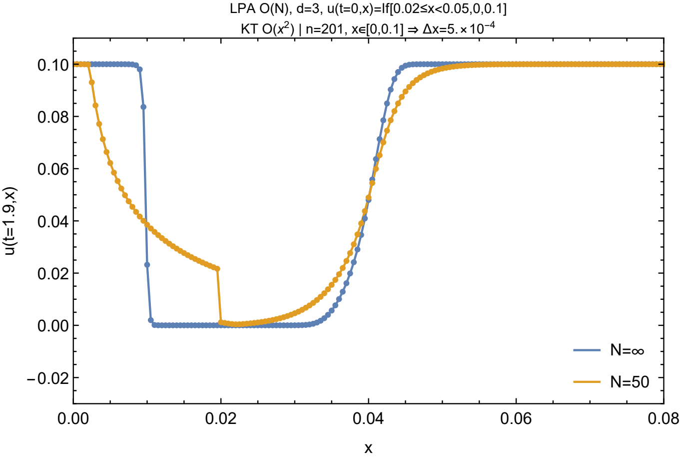
O(N), d=3, (GW.2a) ICs



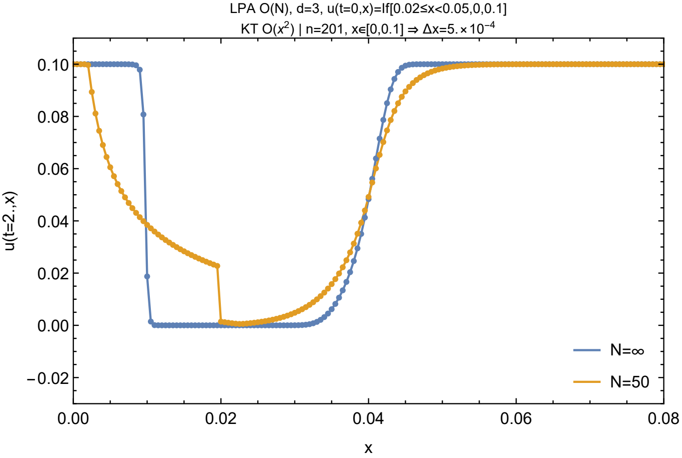
O(N), d=3, (GW.2a) ICs



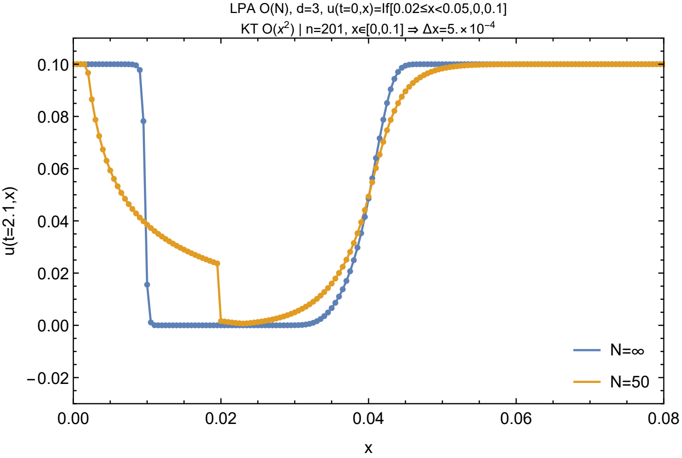
O(N), d=3, (GW.2a) ICs



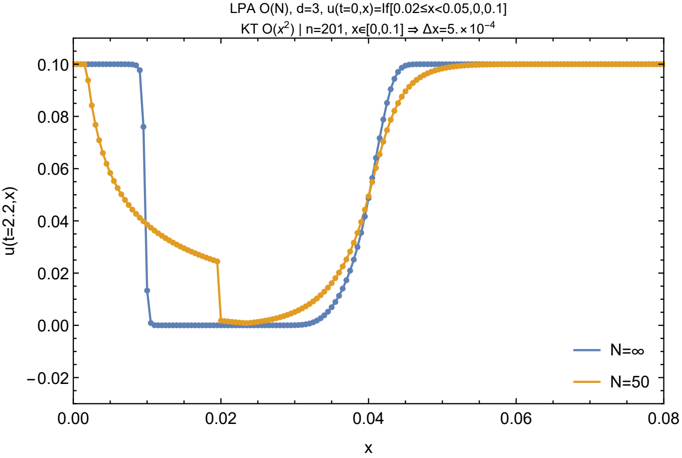
O(N), d=3, (GW.2a) ICs



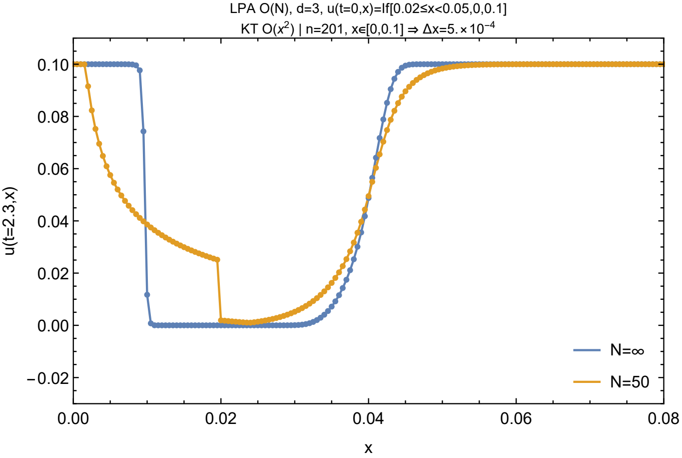
O(N), d=3, (GW.2a) ICs



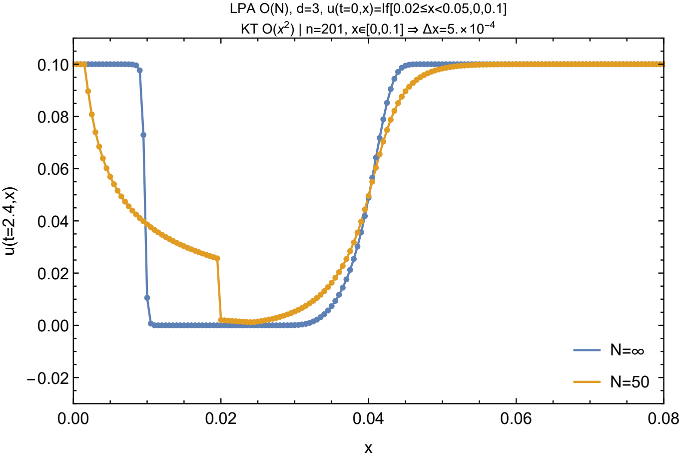
O(N), d=3, (GW.2a) ICs



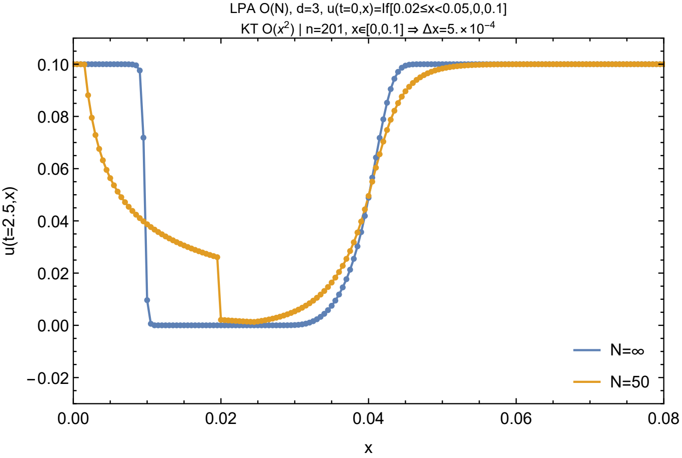
O(N), d=3, (GW.2a) ICs



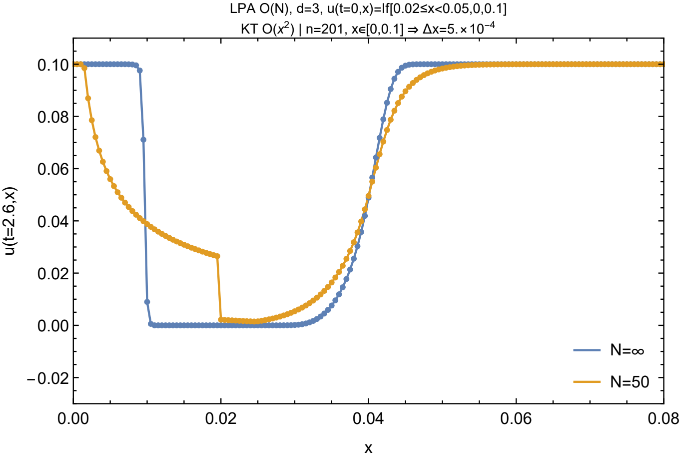
O(N), d=3, (GW.2a) ICs



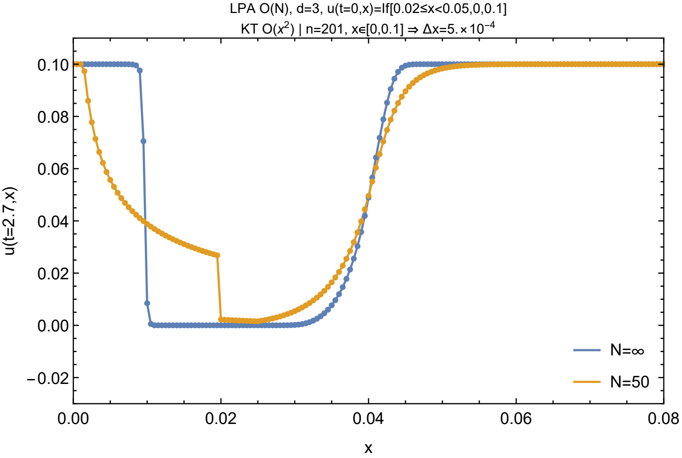
O(N), d=3, (GW.2a) ICs



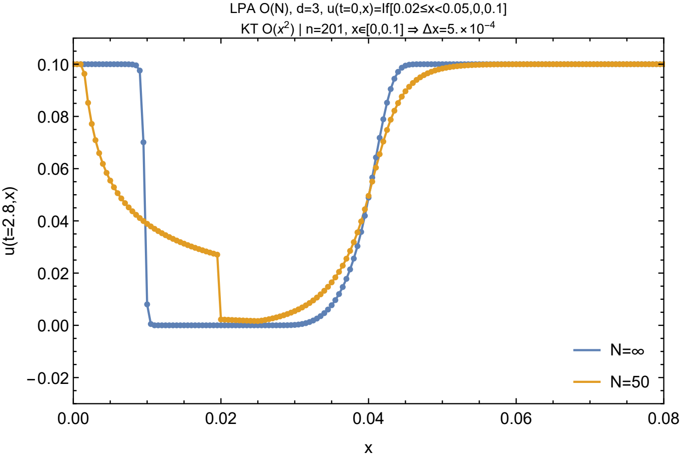
O(N), d=3, (GW.2a) ICs



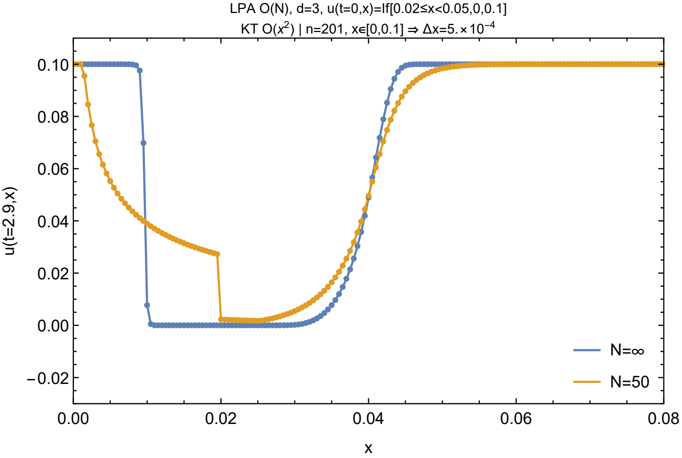
O(N), d=3, (GW.2a) ICs



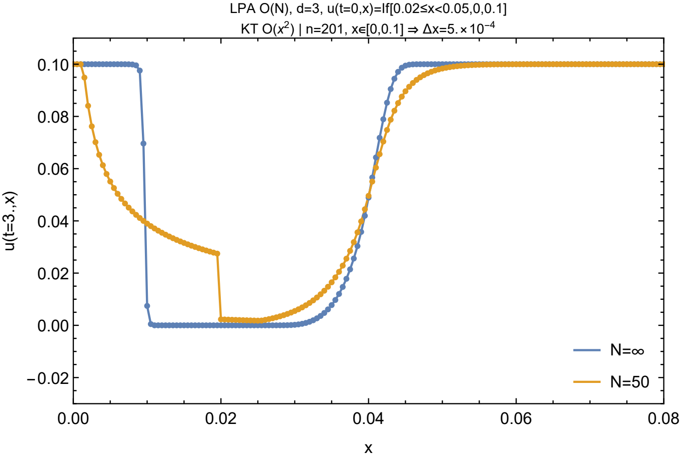
O(N), d=3, (GW.2a) ICs



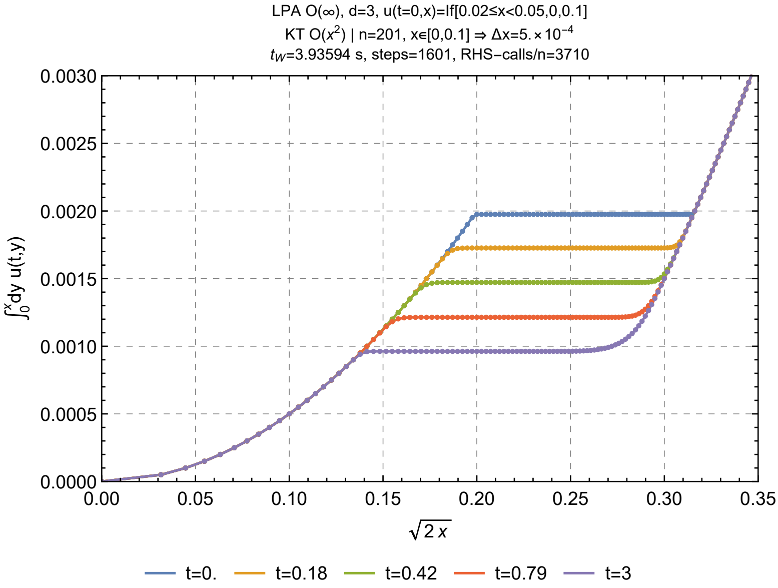
O(N), d=3, (GW.2a) ICs



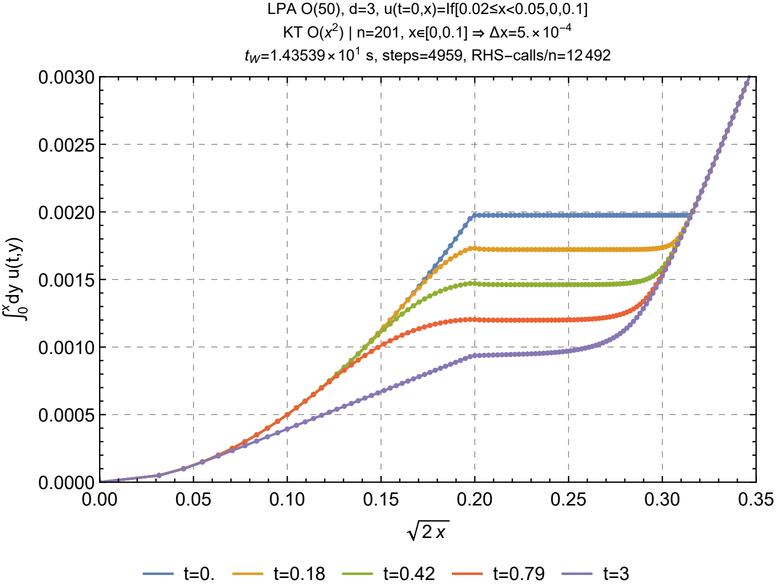
O(N), d=3, (GW.2a) ICs



$O(\infty)$, d=3, (GW.2a)

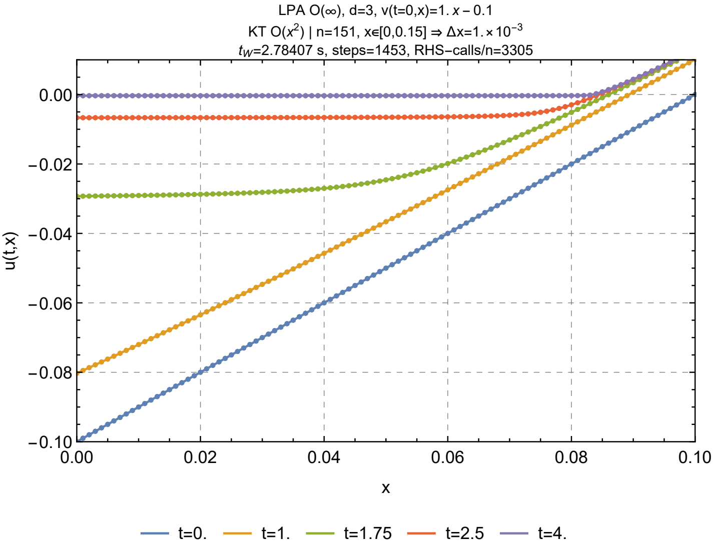


O(50), d=3, (GW.2a)



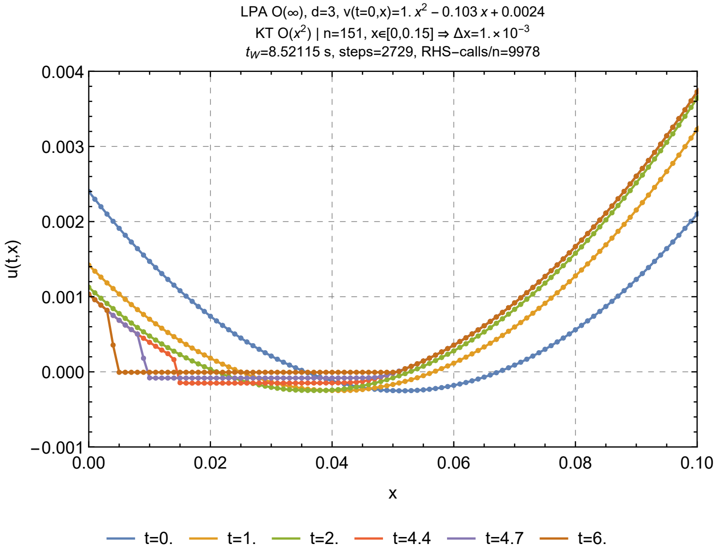
LPA O(50), d=3, u(t=0,x)=If[0.02≤x<0.05,0,0.1]
KT O(x²) | n=201, x∈[0,0.1] ⇒ Δx=5.×10⁻⁴
t_W=1.43539×10¹ s, steps=4959, RHS−calls/n=12492
0.0030
0.0025
0.0020
0.0015
0.0010
0.0005
0.0000
∫₀ˣdy u(t,y)
0.00
0.05
0.10
0.15
0.20
0.25
0.30
0.35
√2x
t=0.
t=0.18
t=0.42
t=0.79
t=3

O(∞), d=3, (GW.3a) ICs

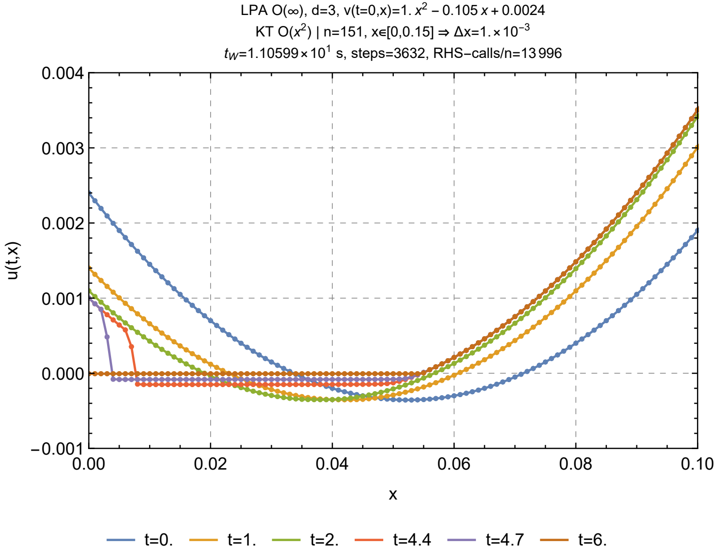


LPA O(∞), d=3, v(t=0,x)=1. x − 0.1
KT O(x²) | n=151, x∈[0,0.15] ⇒ Δx=1.×10⁻³
t_W=2.78407 s, steps=1453, RHS−calls/n=3305
u(t,x)
x
0.00
−0.02
−0.04
−0.06
−0.08
−0.10
0.00
0.02
0.04
0.06
0.08
0.10
t=0.
t=1.
t=1.75
t=2.5
t=4.

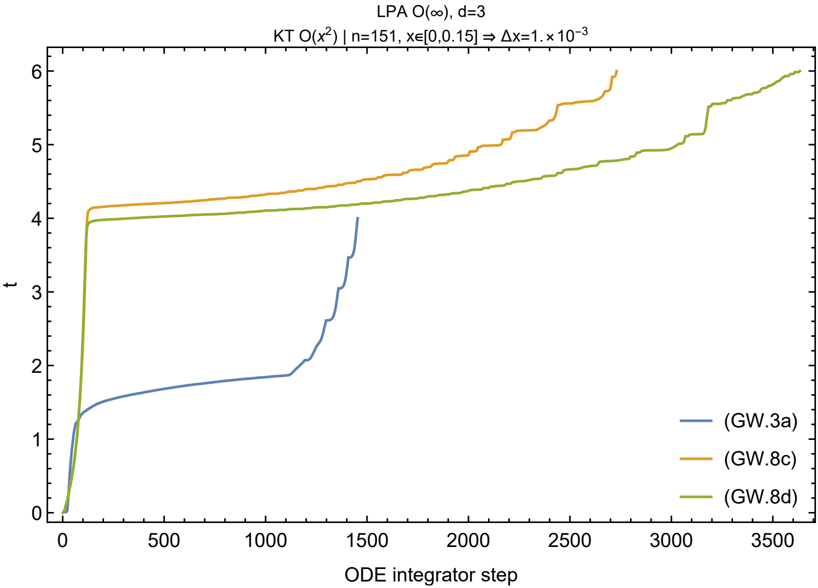
$O(\infty)$, d=3, (GW.8c) ICs



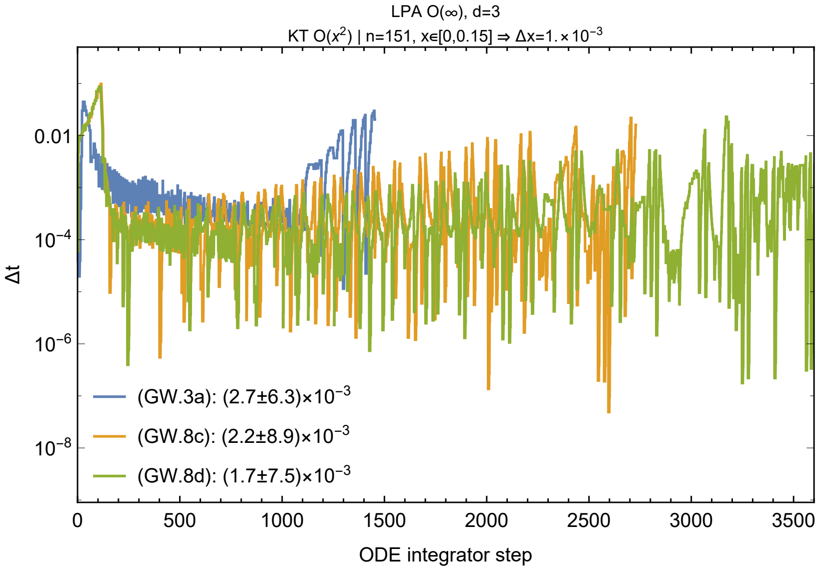
$O(\infty)$, d=3, (GW.8d) ICs



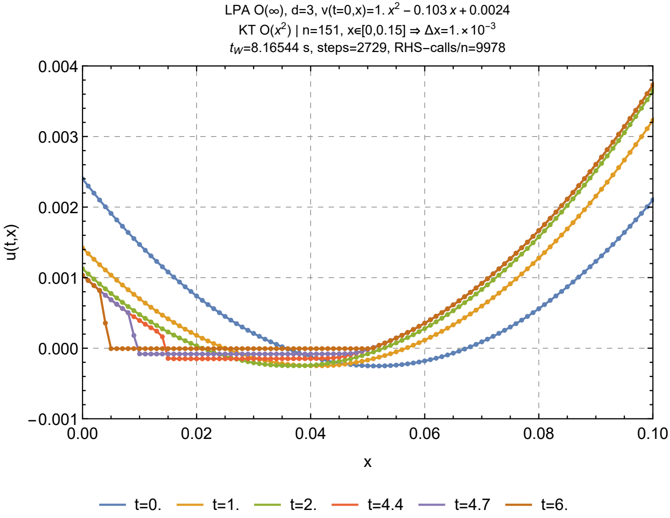
O(∞), d=3 - ODE flow time steps



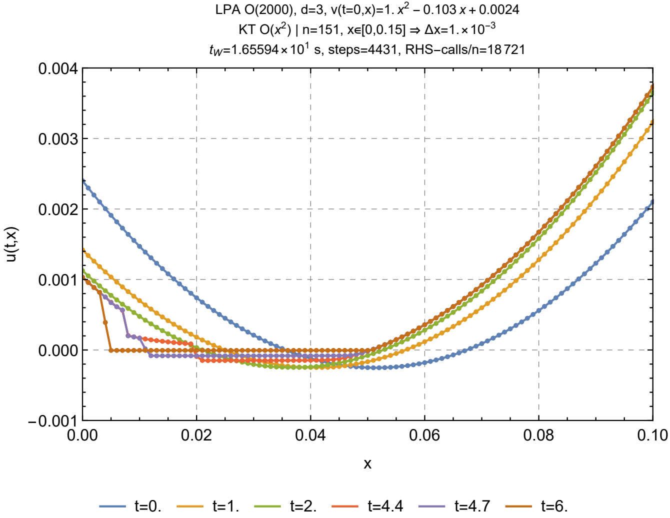
O(∞), d=3 - ODE flow time steps



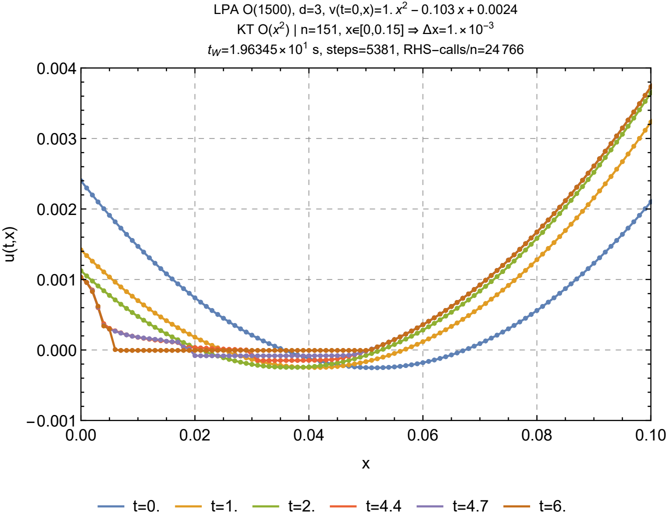
O(N), d=3, (GW.8c) ICs



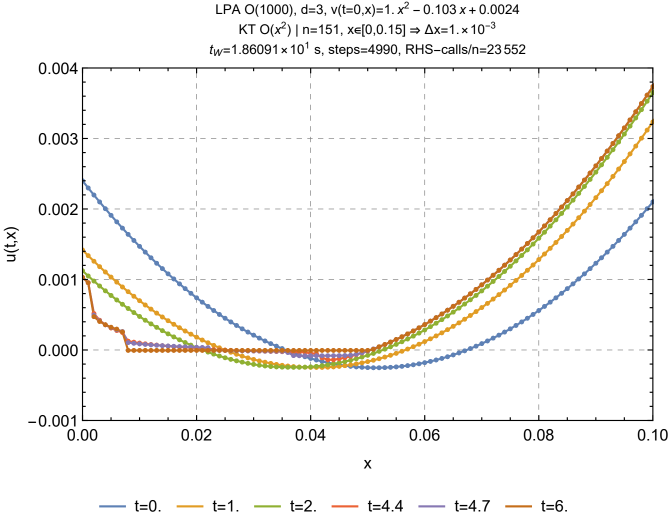
O(N), d=3, (GW.8c) ICs



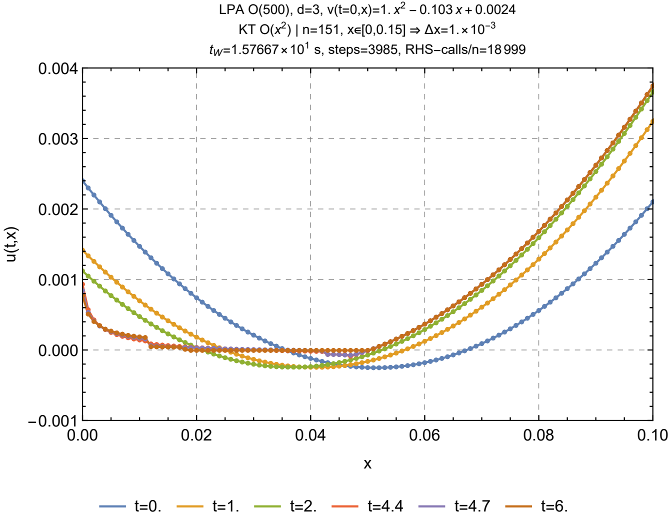
O(N), d=3, (GW.8c) ICs



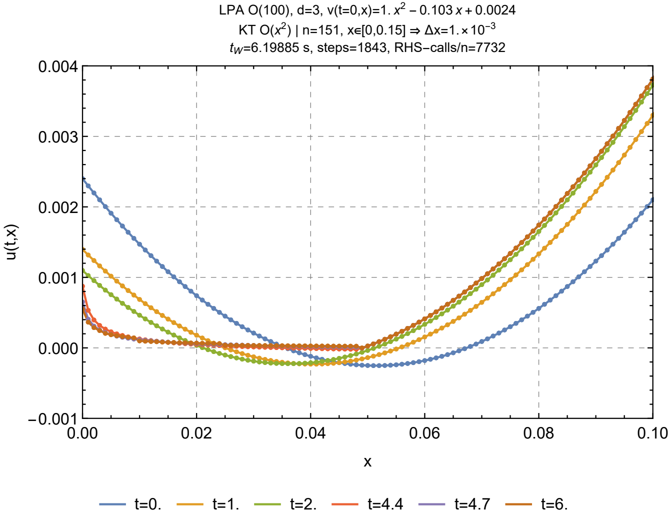
O(N), d=3, (GW.8c) ICs



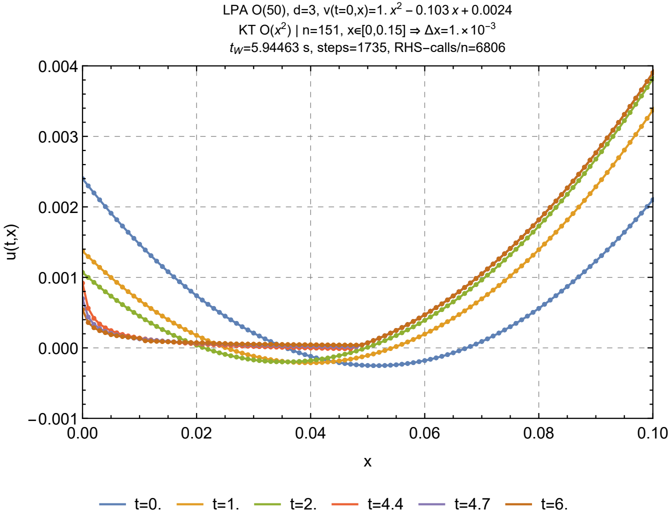
O(N), d=3, (GW.8c) ICs



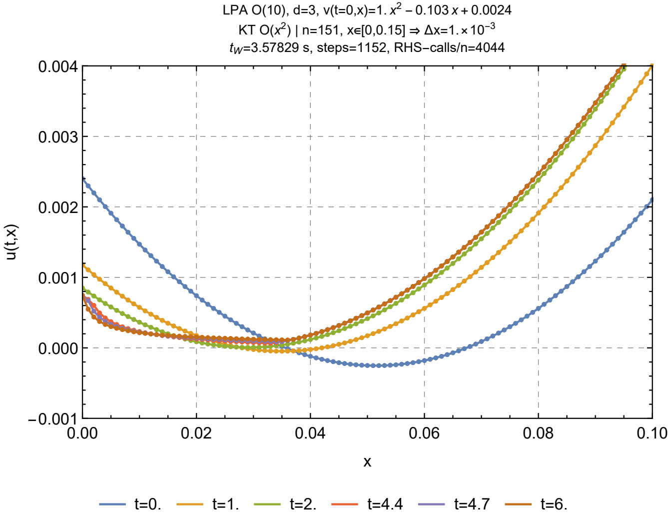
O(N), d=3, (GW.8c) ICs



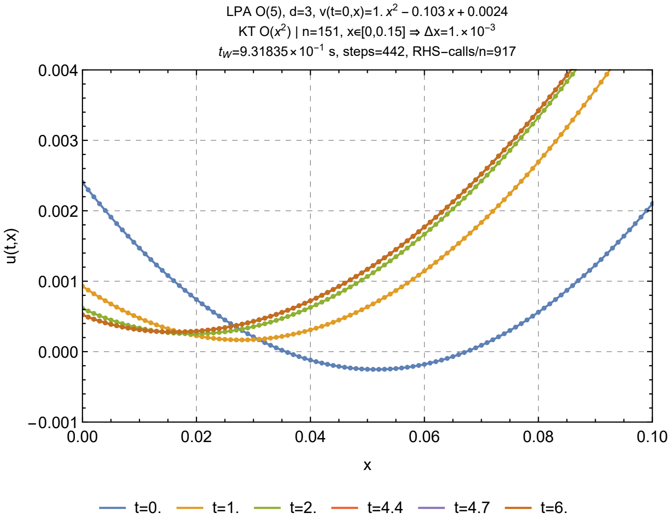
O(N), d=3, (GW.8c) ICs



O(N), d=3, (GW.8c) ICs



O(N), d=3, (GW.8c) ICs



O(N) model in d=0 - Exact solution

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- A zero dimensional ~~field~~-theory of N scalars ϕ_a:

$$S = U(\phi_a\phi^a/2) = \frac{\lambda_1}{2}\phi_a\phi^a + \frac{\lambda_2}{8}(\phi_a\phi^a)^2 + \frac{\lambda_3}{24}(\phi_a\phi^a)^3 + \ldots \quad (17)$$

$$= U(\rho), \qquad \rho \equiv \phi_a\phi^a/2 \quad (18)$$

  - The physical minimum is always at $< \phi_a\phi^a >= 0$
  - All n-Point functions can be computed by solving one-dimensional integrals: e.g.:

$$\Gamma^{(2)} = \frac{NR_{N-1}}{R_{N+1}}, \quad R_N = \int_0^\infty \mathrm{d}y\, y^N \exp\left[-U(y^2/2)\right] \quad (19)$$

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for further details see e.g.: J. Keitel and L. Bartosch, J. Phys. **A45** (2012), arXiv: `1109.3013 [cond-mat.stat-mech]`

O(N) model in d=0 - FRG approach

- The Wetterich eq. in $d = 0$ is a partial differential equation:

$$\frac{\mathrm{d}U_t(\rho)}{\mathrm{d}t} = \frac{N-1}{R_t + U_t'(\rho)}\frac{\mathrm{d}R_t}{\mathrm{d}t} + \frac{1}{R_t + U_t'(\rho) + 2\rho U_t''(\rho)}\frac{\mathrm{d}R_t}{\mathrm{d}t} \tag{20}$$

with $t = 0$ and $R_0 = \infty$ in the UV and $t = 1$ and $R_1 = 0$ in the IR.

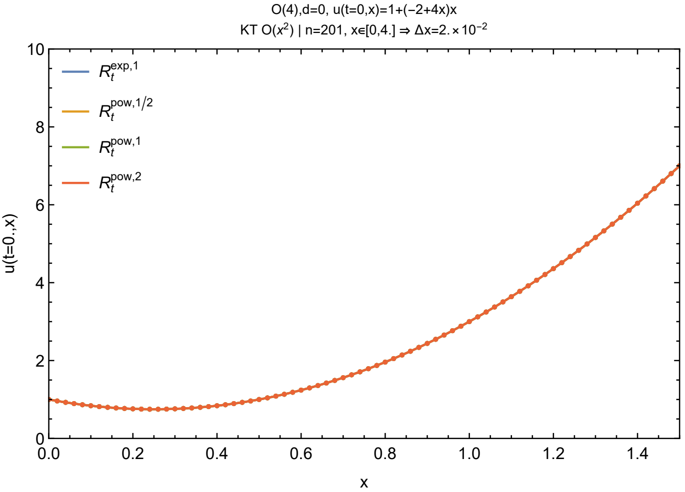
  - Various regulators can be constructed e.g.:

$$R_t^{\mathrm{exp},m} = \exp[1/t^m - 1] - 1 \quad \text{with } m > 0 \tag{21}$$

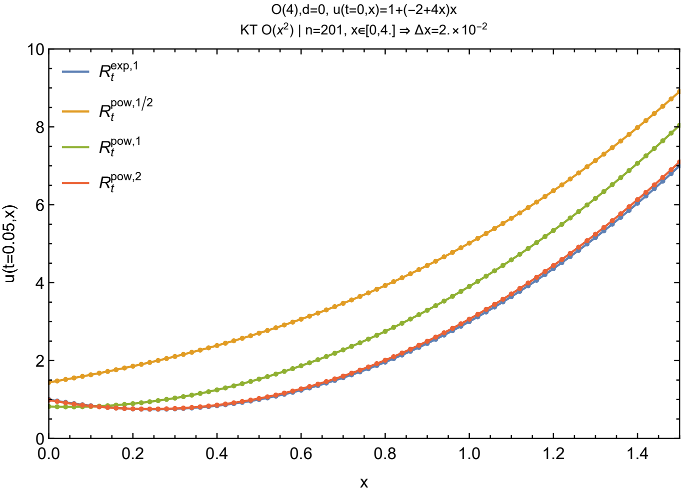
$$R_t^{\mathrm{pow},m} = 1/t^m - 1, \quad \text{with } m > 0 \tag{22}$$

  - The exact flow eqs. for $\partial_\rho U_t$ can be solved numerically and results can be confronted with n-point functions form exact integration

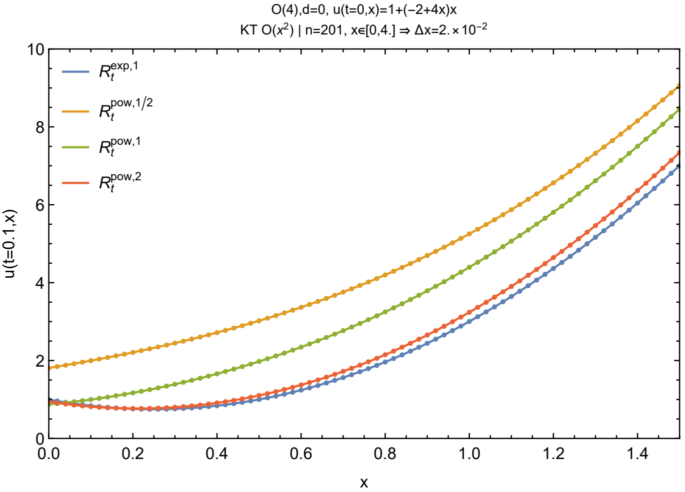
O(N) model in d=0 - Regulator trajectories



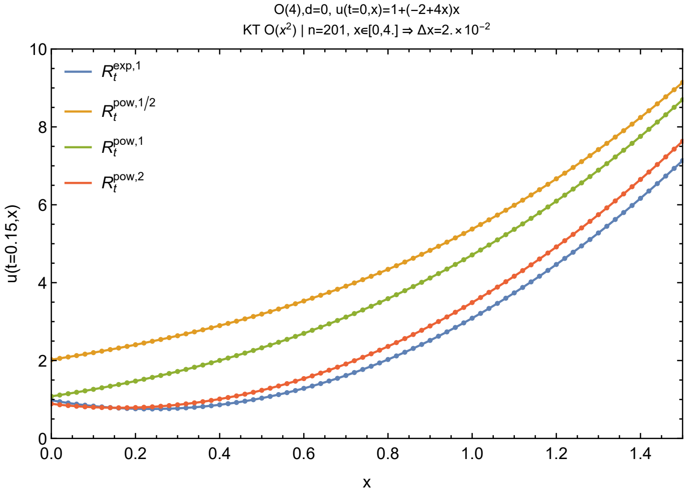
O(N) model in d=0 - Regulator trajectories



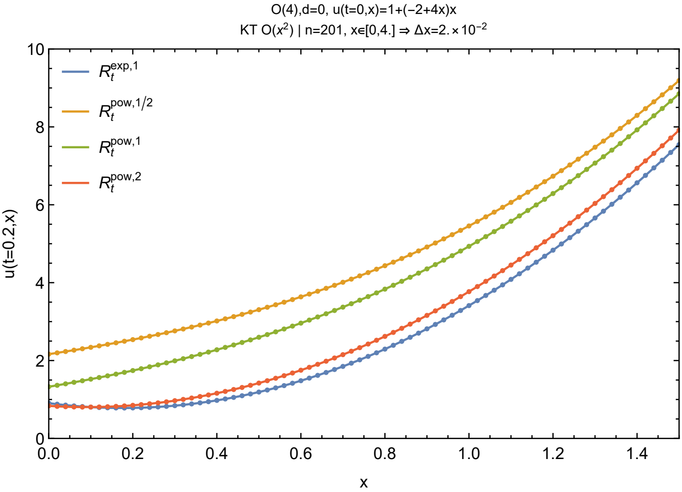
O(N) model in d=0 - Regulator trajectories

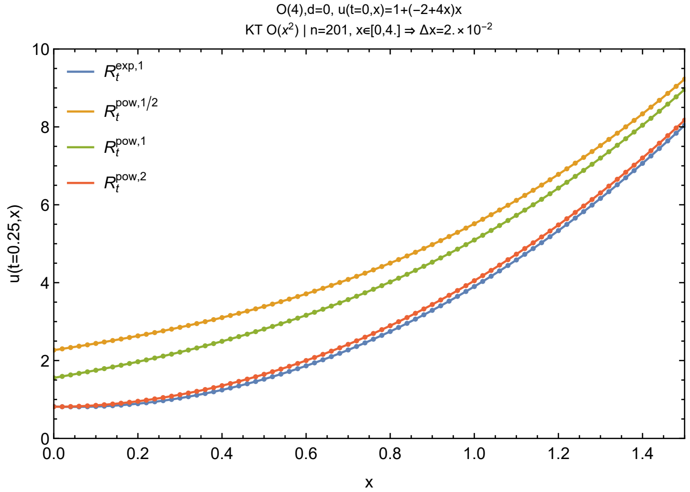


O(N) model in d=0 - Regulator trajectories



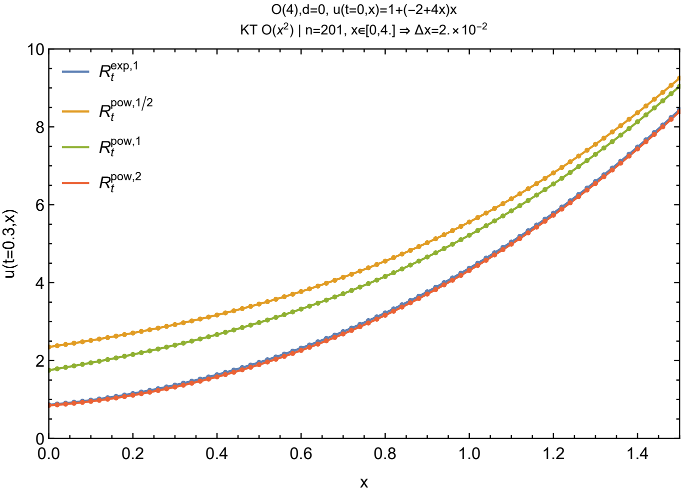
O(N) model in d=0 - Regulator trajectories





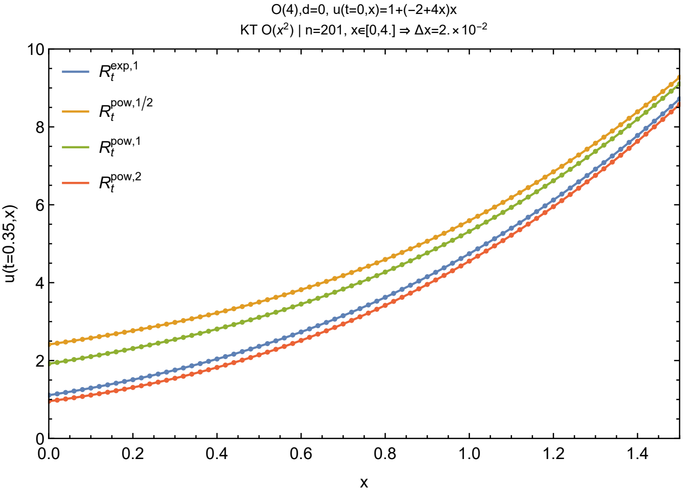
O(4),d=0, u(t=0,x)=1+(−2+4x)x
KT O(x²) | n=201, x∊[0,4.] ⇒ Δx=2.×10⁻²
$R_t^{exp,1}$
$R_t^{pow,1/2}$
$R_t^{pow,1}$
$R_t^{pow,2}$
u(t=0.25,x)
x
0
2
4
6
8
10
0.0
0.2
0.4
0.6
0.8
1.0
1.2
1.4

O(N) model in d=0 - Regulator trajectories

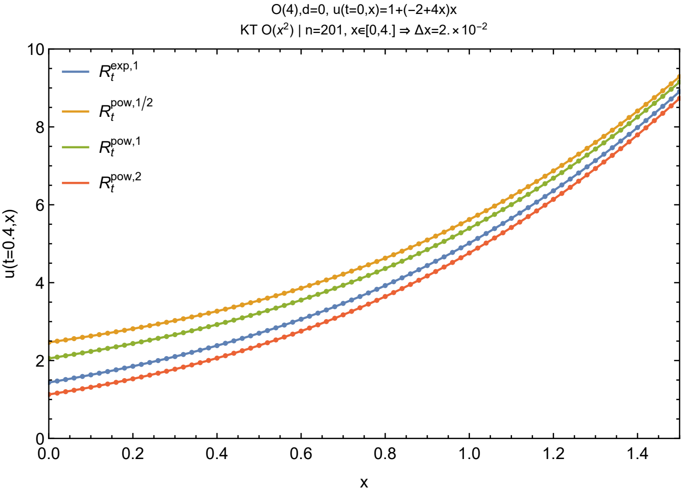


O(N) model in d=0 - Regulator trajectories

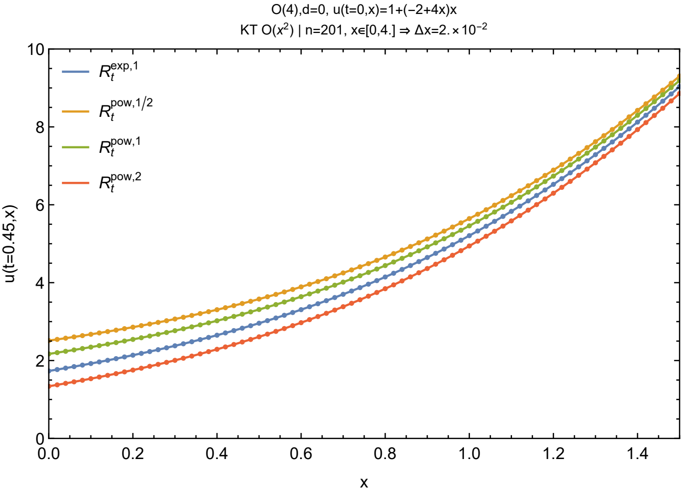
CRC-TR 211



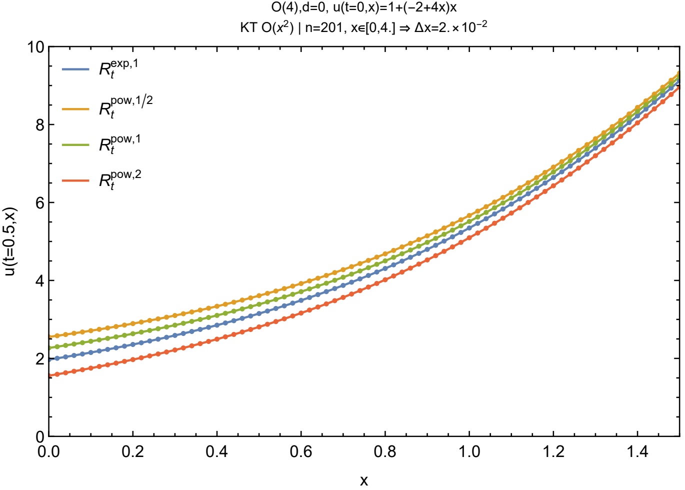
O(N) model in d=0 - Regulator trajectories



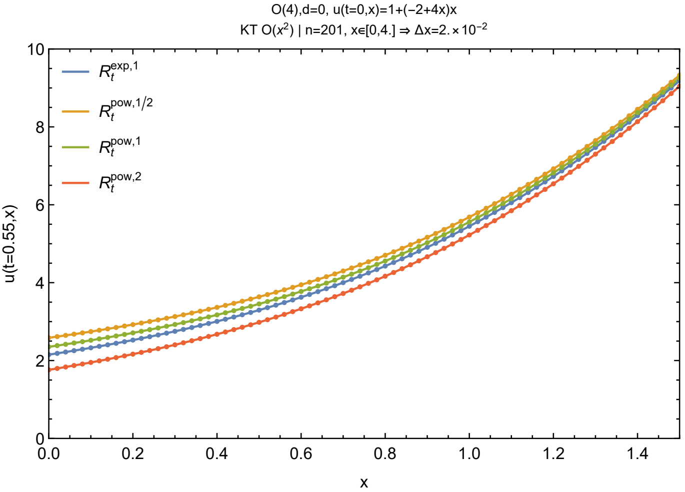
O(N) model in d=0 - Regulator trajectories



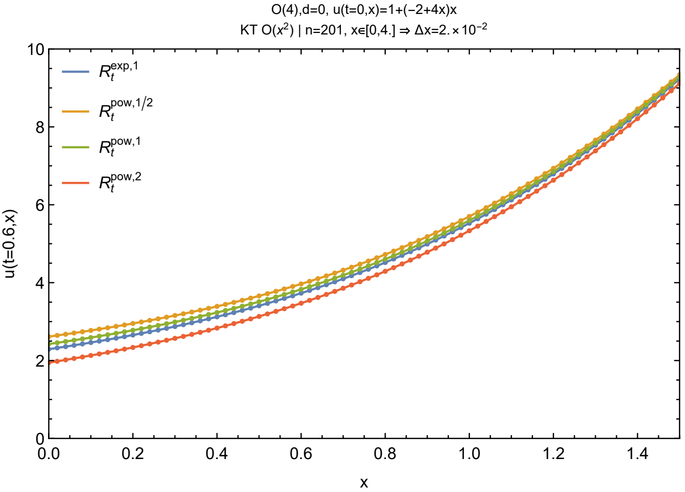
O(N) model in d=0 - Regulator trajectories



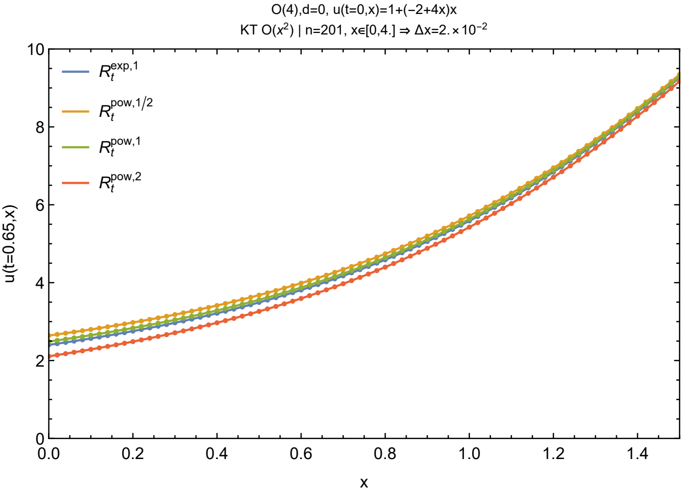
O(N) model in d=0 - Regulator trajectories



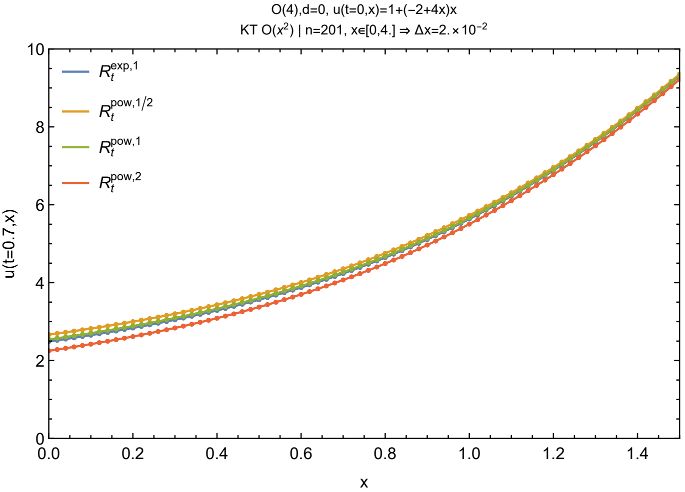
O(N) model in d=0 - Regulator trajectories



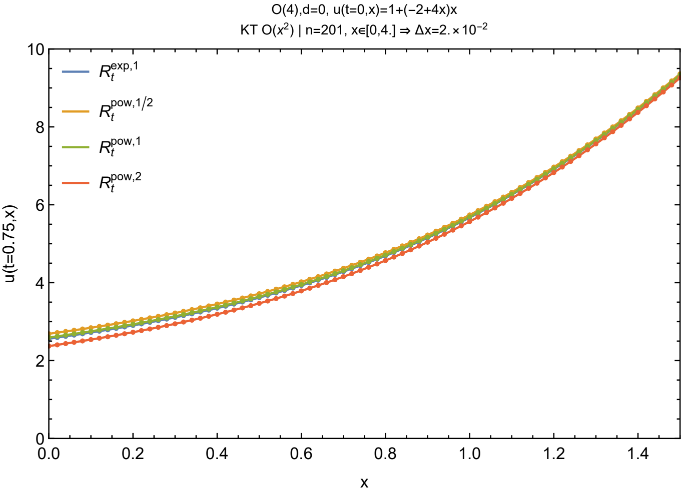
O(N) model in d=0 - Regulator trajectories



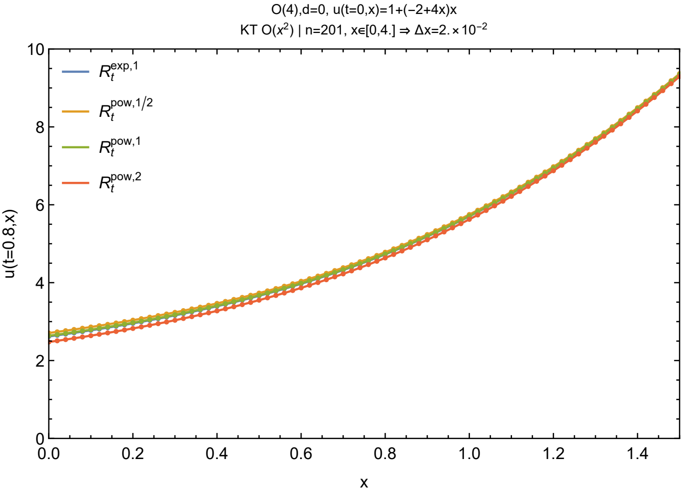
O(N) model in d=0 - Regulator trajectories



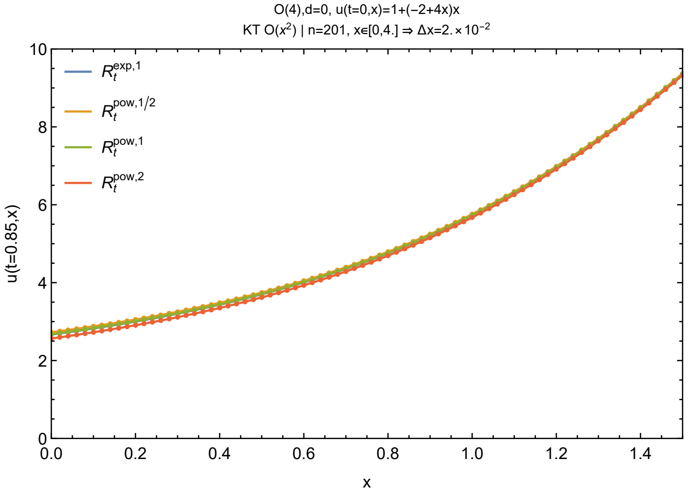
O(N) model in d=0 - Regulator trajectories



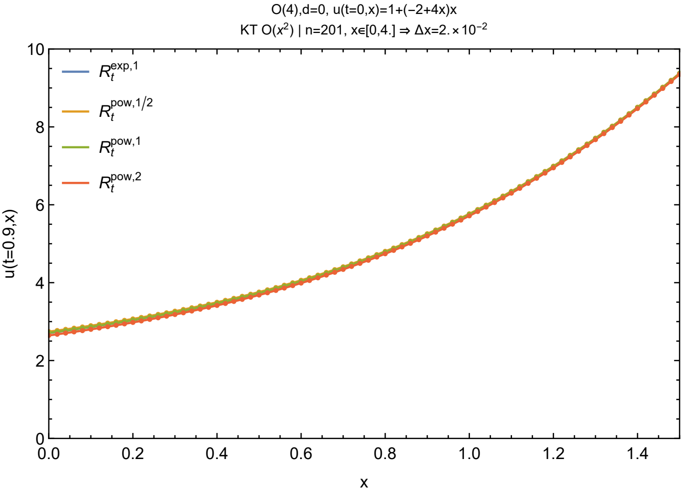
O(N) model in d=0 - Regulator trajectories



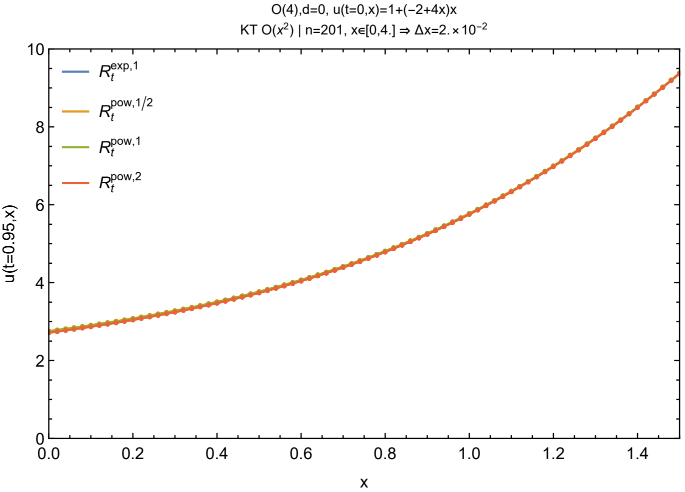
O(N) model in d=0 - Regulator trajectories



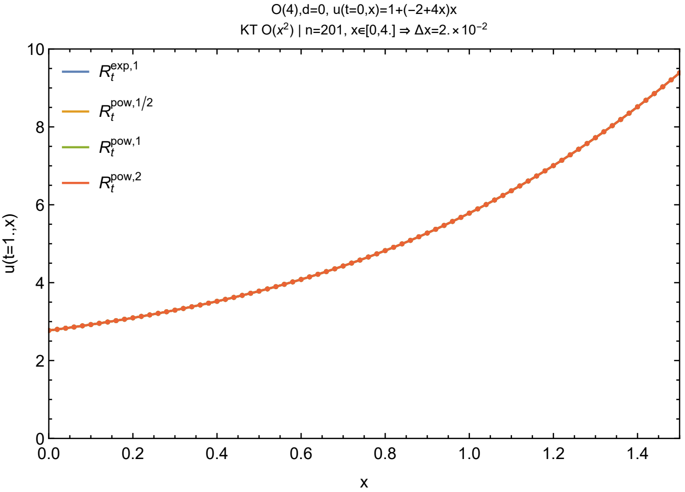
O(N) model in d=0 - Regulator trajectories



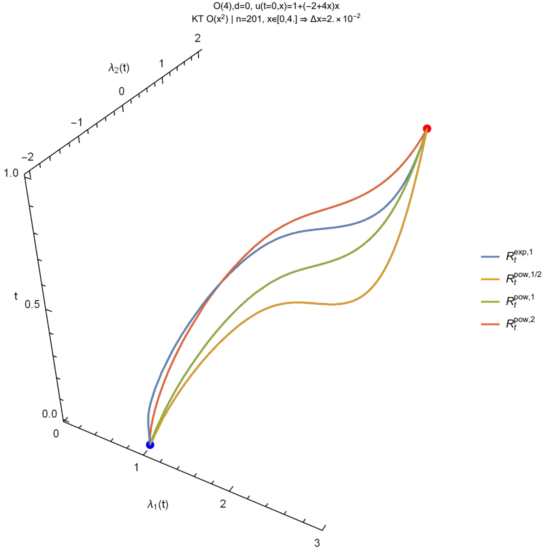
O(N) model in d=0 - Regulator trajectories



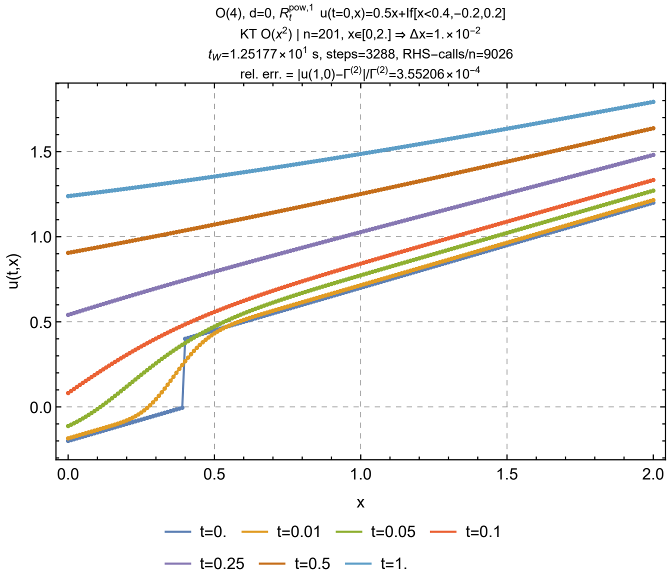
O(N) model in d=0 - Regulator trajectories



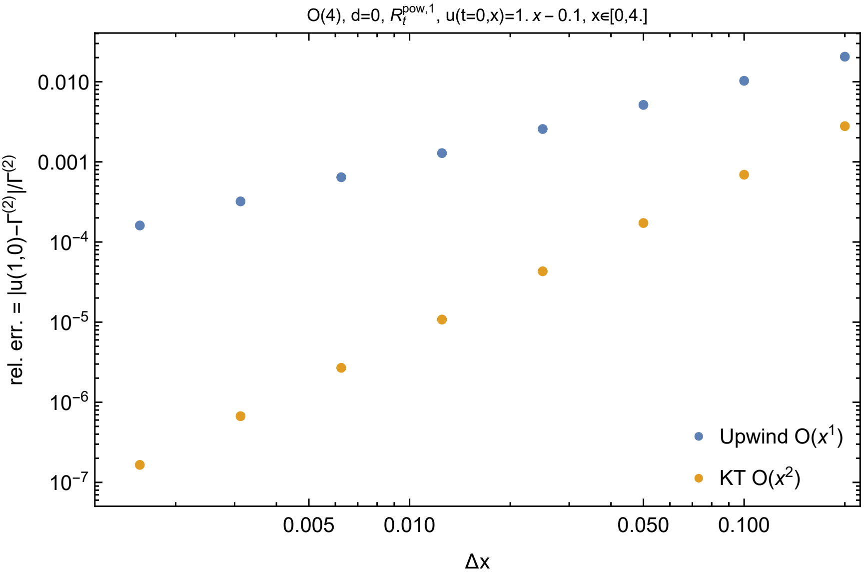
O(N) model in d=0 - $\lambda_1(t)$ and $\lambda_2(t)$



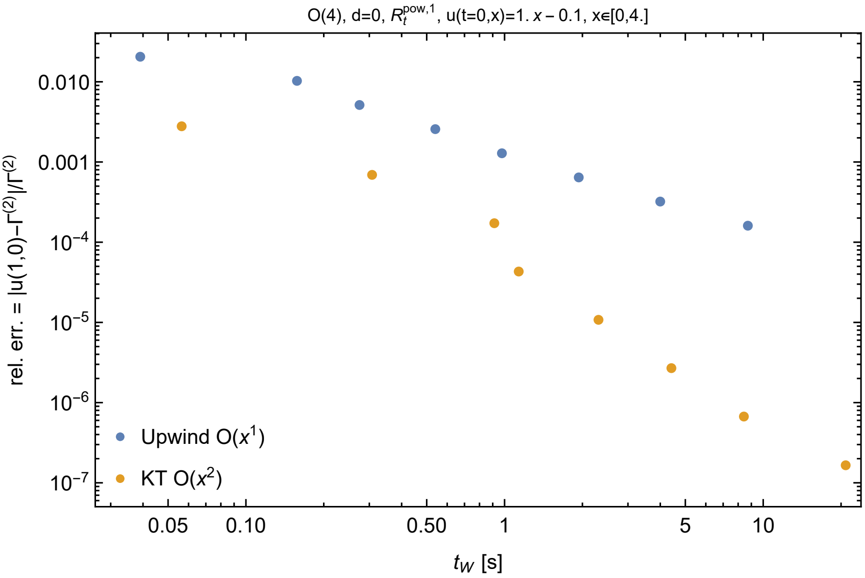
O(4) model in d=0 - Discontinuous IC



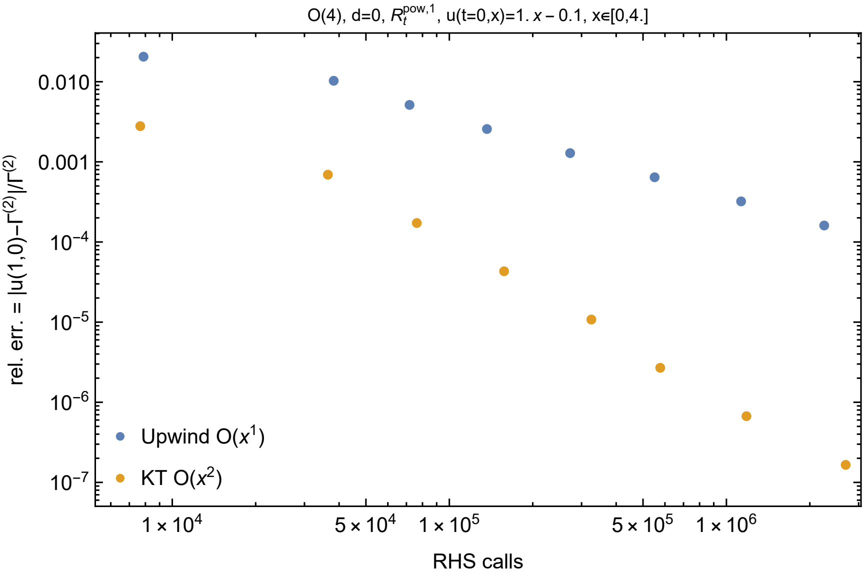
O(4) model in d=0 - Upwind O(x^1) vs KT O(x^2)



O(4) model in d=0 - Upwind $O(x^1)$ vs KT $O(x^2)$



O(4) model in d=0 - Upwind O(x^1) vs KT O(x^2)



Summary and Outlook

- The second order KT-scheme in its semi-discrete form (4)
  - Robust and competitive spatial discretisation scheme
  - Easy to implement/extend
  - First applications to FRG LPA flow eqs. look very promising
- O(N) model in zero dimensions
  - Well suited for numerical test (exact solutions, non-trivial flow eqs.)
  - Very instructive toy model for various aspects of FRG and QFTs in general
- **TODO:**
  - Finite volume/KT scheme beyond LPA
  - Finite volume/KT scheme at finite T and μ
  - ...