Numerical fluid dynamics for FRG flow equations: Zero-dimensional QFTs as numerical test cases

Introduction

A natural approach to test the applicability of a method is to study problems with exact reference solutions. The calculation of correlation functions of QFTs in $0+0$ spacetime dimensions reduces to the calculation of ordinary integrals, which can be evaluated to extremely high precision, providing excellent reference values. Consequently, $(0+0)$-dimensional QFTs are perfectly suited as testing grounds for (non-)perturbative methods and numerical tools in QFT. Furthermore, they can provide a low-level introduction to novel methods for non-experts or students.

In our work, we use the $(0+0)$-dimensional $O(N)$-model to demonstrate the analogy between FRG-flow equations and non-relativistic classical fluid dynamics. We apply methods from numerical fluid dynamics to FRG-flow equations and test the precision of these methods against exact solutions. Leveraging this framework we study different aspects of RG flows in zero dimensions including truncations, irreversibility and RG-consistency. Most of our findings can be generalized to QFTs in higher dimensions.

The zero-dimensional $O(N)$ model

We study a zero-dimensional QFT of N real scalars with expectation values

$$\langle(\vec{\phi}^{\,2})^n\rangle = \frac{2^n \int_0^\infty \mathrm{d}\rho\, \rho^{\frac{(N-2)}{2}}\, \rho^n\, \mathrm{e}^{-U(\rho)}}{\int_0^\infty \mathrm{d}\rho\, \rho^{\frac{(N-2)}{2}}\, \mathrm{e}^{-U(\rho)}}\,, \qquad \rho = \tfrac{1}{2}\vec{\phi}^{\,2}\,,$$

with the $O(N)$ invariant ρ and self-interaction potential $U(\rho)$. All non-vanishing 1PI-correlation functions $\Gamma^{(2n)}$ can be expressed in terms of $\langle(\vec{\phi}^{\,2})^m\rangle$ with $0 \leq m \leq n$, *cf.* Ref. [4].

FRG and numerical fluid dynamics

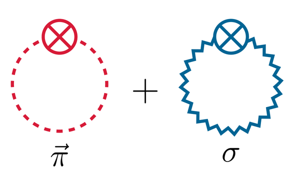
The RG flow eq. of the zero-dimensional $O(N)$ model

In the Wetterich-Ellwanger-Morris formulation the Renormalization Group (RG) flow of the $O(N)$ model is described by the exact RG equation,

$$\partial_t \bar{\Gamma}_t[\vec{\varphi}] = \mathrm{tr}\Big[\big(\tfrac{1}{2}\,\partial_t R_t\big)\,\big(\bar{\Gamma}_t^{(2)}[\vec{\varphi}] + R_t\big)^{-1}\Big]\,,$$

where $\vec{\varphi} = (\sigma, \vec{\pi})$ denotes the $O(N)$-vector of pions and the radial sigma mode. In $0+0$ spacetime dimensions the ERG equation is exact (no truncation needed) and the most general effective average action is given by a scale dependent local potential $\bar{\Gamma}_t[\vec{\varphi}] = U(t, \vec{\varphi})$. The corresponding **exact partial differential equation** reads

$$\partial_t U(t,\sigma) = \frac{(N-1)\,\frac{1}{2}\,\partial_t r(t)}{r(t) + \frac{1}{\sigma}\partial_\sigma U(t,\sigma)} + \frac{\frac{1}{2}\,\partial_t r(t)}{r(t) + \partial_\sigma^2 U(t,\sigma)} = \;\; (\vec{\pi}) \;+\; (\sigma)\,,$$



where $r(t) = \Lambda\,\mathrm{e}^{-t}$ denotes the regulator with the UV initial scale Λ and $t \in [0, \infty)$ positive RG time (note the sign convention).

Numerical methods for flow equations

The above flow equation can be reformulated as a non-linear partial differential equation in RG time t and field space σ for its derivative $u(t,\sigma) = \partial_\sigma U(t,\sigma)$. This PDE is an **advection-diffusion equation** [3, 1], where the field space $x \equiv \sigma$ plays the role of an effective spatial domain,

$$\partial_t u(t,x) + \partial_x \underbrace{F[t,x,u(t,x)]}_{\vec{\pi}\,\sim\,\text{advection}} = \partial_x \underbrace{Q[t,\partial_x u(t,x)]}_{\sigma\,\sim\,\text{diffusion}}\,,$$

The formulation as a conservation equation strongly suggests the use of established discretization schemes from numerical fluid dynamics for the numerical solution. For its robustness and relative simplicity we decided to use a **finite volume method** – the Kurganov-Tadmor central scheme [2].

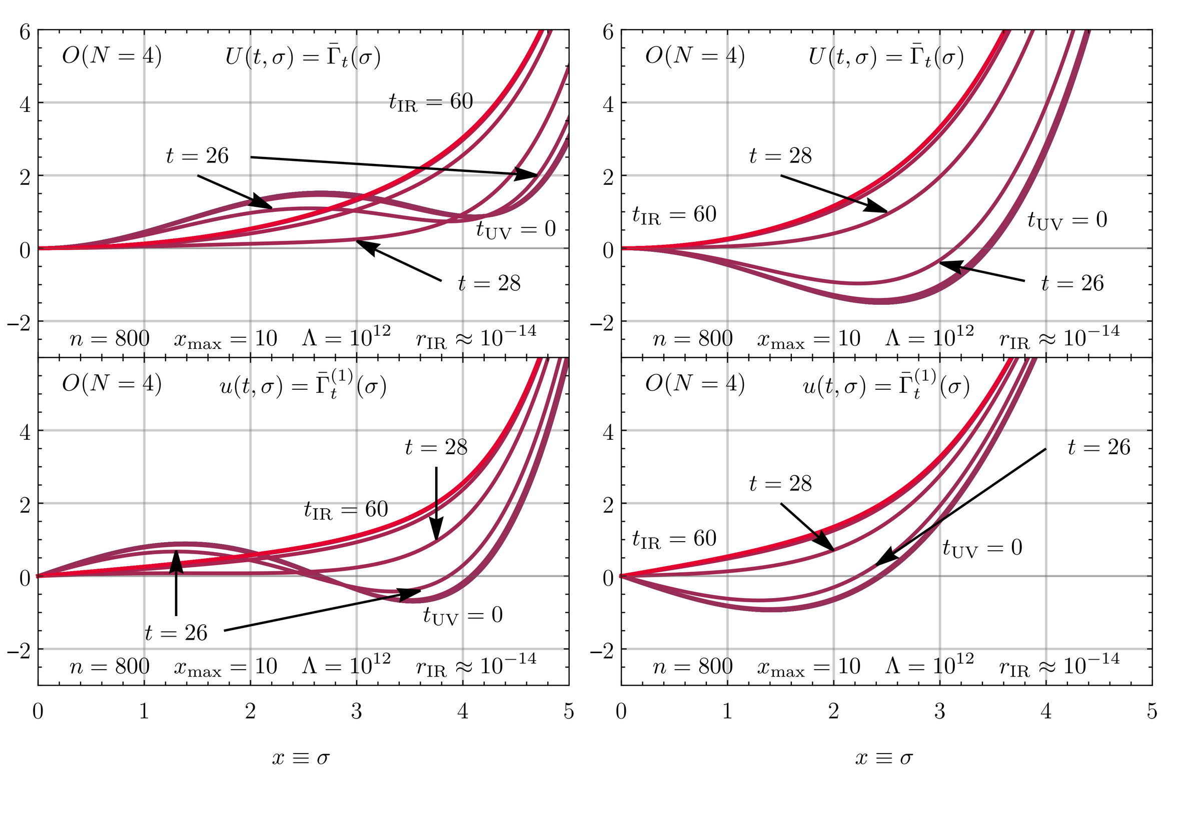


Figure 1: The plots show the RG flow of the effective potential $U(t,\sigma)$ (upper panels) and its derivative $u(t,\sigma) = \partial_\sigma U(t,\sigma)$ (lower panels) for the zero-dimensional $O(N=4)$-model. In the left column we study the flow of a ϕ^6-potential with a local minimum in the UV. In the right column we consider a ϕ^4 model with a negative mass term.

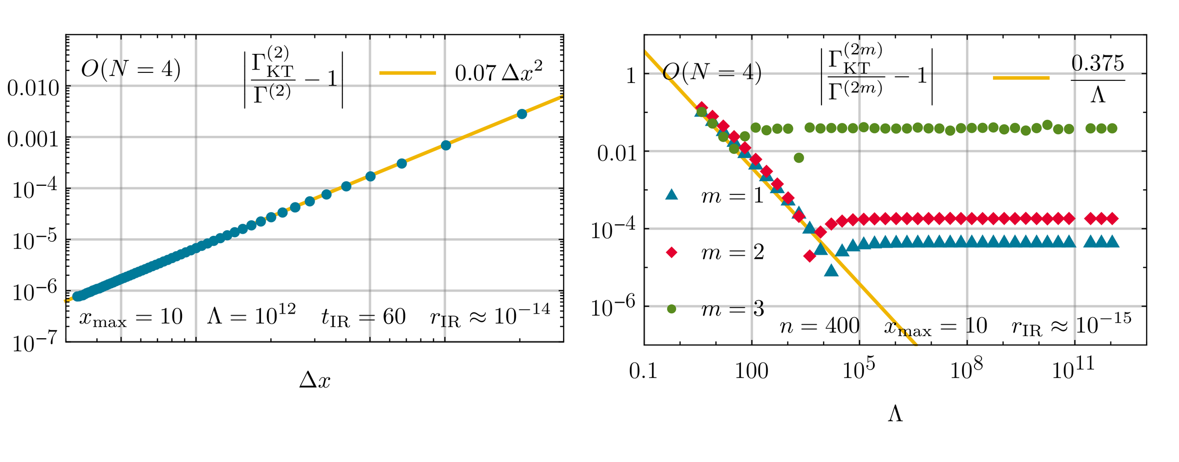


Figure 2: The plots show the relative errors of selected 1PI-correlation functions $\Gamma^{(2n)}$ in the IR for the $O(N=4)$-model with the quartic potential and negative mass term from Fig. 1 over spatial resolution Δx (left panel) and UV inital scale Λ (right panel).

The FRG Taylor expansion

Zero-dimensional models can be used to test truncation schemes in FRG. A common truncation scheme for the effective potential is the Taylor/vertex expansion,

$$\bar{\Gamma}_t[\varrho] = \sum_{n=0}^{m} \frac{\bar{\Gamma}^{(2n)}(t)}{(2n-1)!!}\,\frac{\varrho^n}{n!}\,. \qquad \varrho \equiv \tfrac{1}{2}\,\vec{\varphi}^{\,2}\,,$$

where the RG flow is calculated on a finite set of m ODEs for the vertices $\bar{\Gamma}^{(2n)}(t)$. For the zero dimensional $O(N)$ model the applicability of the FRG Taylor/vertex expansion of the potential is very limited: it is **not applicable to non-analytic potentials** and **fails to converge for non-convex analytic UV potentials**, *e.g.*, ϕ^4 and ϕ^6 potentials.

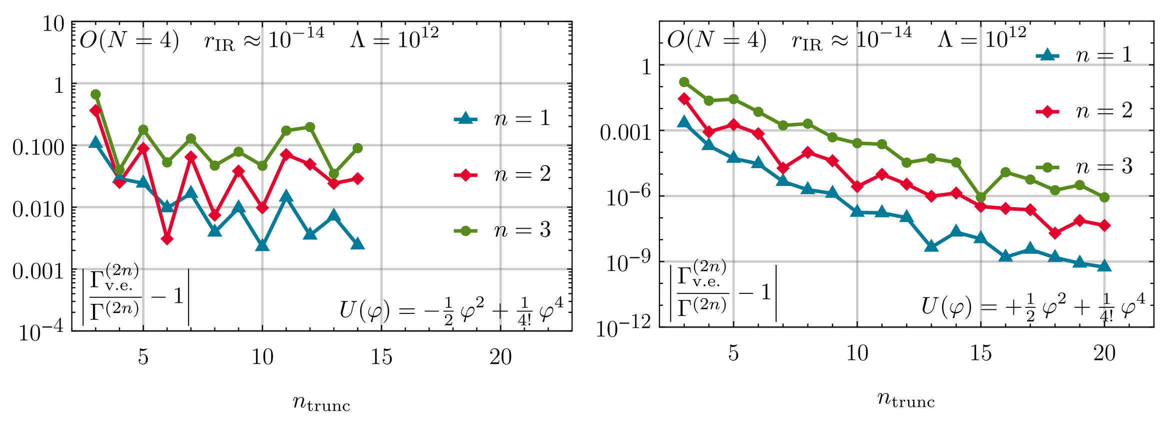


Figure 3: Relative errors of $\Gamma^{(2n)}$ in the IR for $n=1,2,3$ and $O(N=4)$, which were calculated via the RG flow of the FRG Taylor (vertex) expansion to order $m = 2n_{\mathrm{trunc}}$. As initial condition we use the same ϕ^4-potential as in Fig. 1 with negative mass term (left panel) or positive mass term (right panel).

Entropy production and irreversibility of RG flows

The formulation of FRG flow equations as advection diffusion equations naturally leads to a notion of (numerical) entropy directly related to irreversibility of RG flows [1].

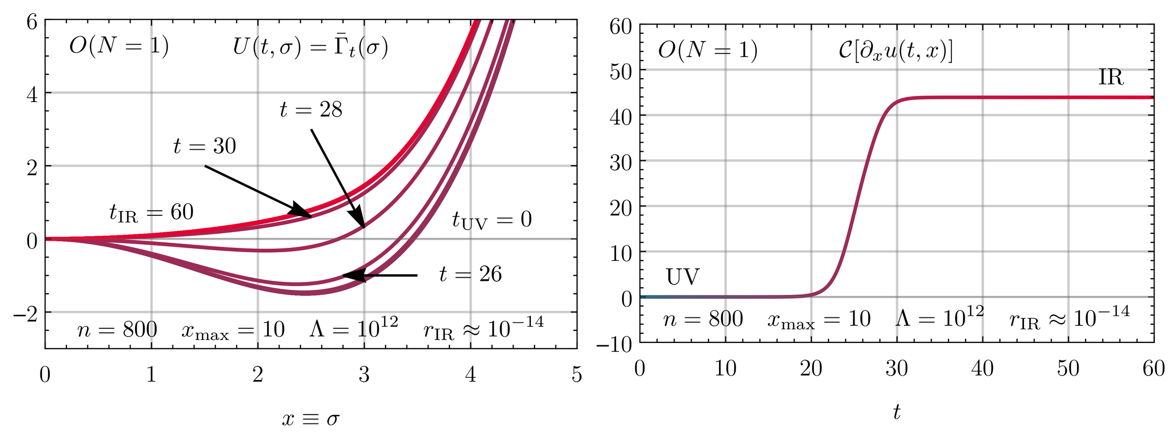


Figure 4: The plots show the RG flow of the effective potential $U(t,\sigma)$ (left panel) and the corresponding entropy production (right panel) for the zero-dimensional $O(N=1)$-model for a ϕ^4-potential with negative mass term.

Finite vs. infinite N – shocks and rarefaction waves in RG flows

A study of the $O(N)$ model within the FRG setup reveals fundamental qualitative differences between RG flows at finite and infinite N. The absence of diffusive contributions at infinite N allows for **non-smooth** and **non-convex** rescaled potentials in the IR, which is a violation of the Mermin-Wagner theorem. The underlying dynamics of the RG flow can be understood using established concepts for non-linear advection equations like the notions of shocks and rarefaction waves [3, 1].

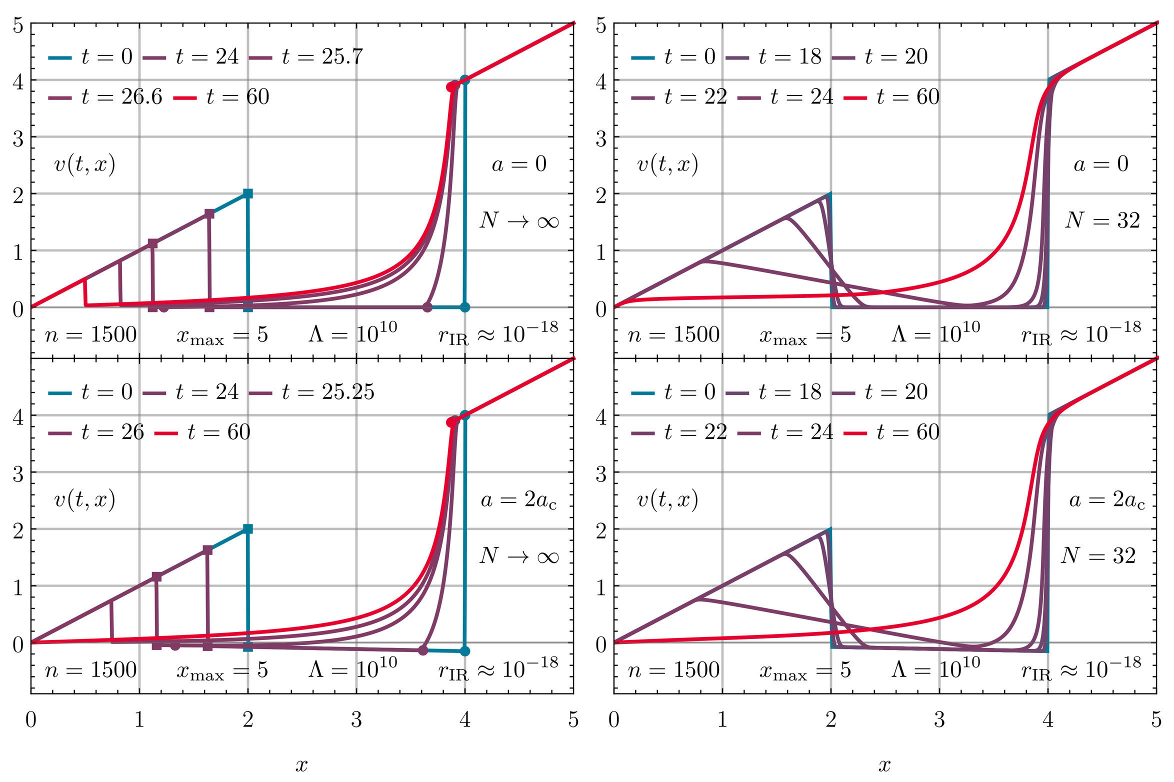


Figure 5: The plots show RG flows of the derivative $v(t,x) \equiv \partial_x V(t,x)$ of a rescaled ($x \mapsto x/\sqrt{N}$ and $U(t,x) \mapsto V(t,x) \equiv U(t,x)/\sqrt{N}$) potential $V(t,x)$ for a family of specific piecewise quadratic UV initial conditions $V(t=0,x)$. At infinite N (left panel) small changes of the UV initial condition lead to distinct results in the IR due to interactions of shock and rarefaction waves. At finite $N=32$ (right panel) diffusive contributions of the σ-mode lead to a qualitatively distinct dynamic compared to the flow in the limit $N \to \infty$.

Summary

- RG flows in the LPA are advection-diffusion(-sink) equations.
- RG flows produce (numerical) entropy directly related to the irreversibility of RG transformations, which is hard coded in the diffusive character of the flow equations.
- RG flows can involve shock and rarefaction waves – especially at large or infinite N.
- The non-linear diffusion in field space stemming from the radial σ-mode is essential to obtain convex potentials in the IR.

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