Inhomogeneous chiral condensates in QCD-inspired models via the FRG

Introduction

Chiral condensation in medium

A theoretical description of strongly interacting matter from first principles using the theory of quantum chromodynamics (QCD) has proved difficult especially at non-zero chemical potential μ and non-zero temperature T. In the past decades chiral low-energy effective models of QCD have been used to study the phase structure and properties of strongly interacting matter in regimes, where computations from first principles overtax our computational abilities.

Mean-field (MF) studies of such models, including only fermionic quantum fluctuations, predict **inhomogeneous chiral condensation** at low temperatures and moderate densities, see Fig. 1. Those crystalline-like phases have proven rather robust against model-extensions and variations of external parameters. For an extensive review, see [1].

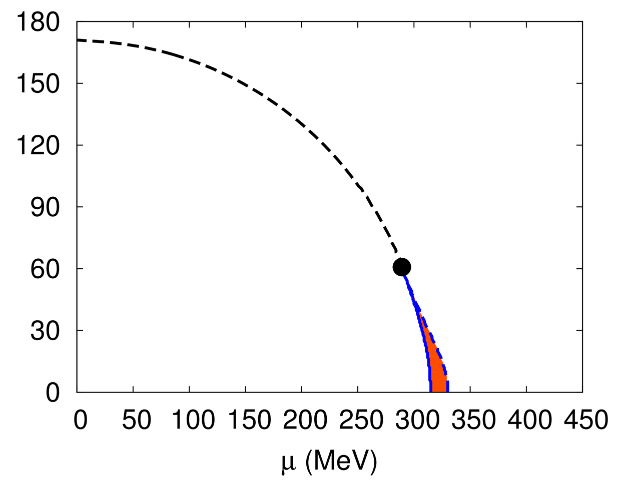


Figure 1: MF phase diagram of the quark-meson model in the chiral limit, allowing for inhomogeneous condensation using Pauli-Villars regularization. The shaded area indicates the inhomogeneous phase with the chiral density wave as the spatial modulation. Figure taken from [2].

Of the several open questions regarding those exotic phases, we are primarily focused on the central question that concerns the stability of inhomogeneous chiral condensation against bosonic quantum fluctuations.

We are facing this question using the **functional renormalization group (FRG)**, which enables us to

- include bosonic/fermionic quantum fluctuations at non-zero μ and T,
- study the stability of inhomogeneous chiral condensation against bosonic quantum fluctuations.

Model and method

Quark-meson model and the FRG

As a chiral low-energy effective model of QCD we consider the two-flavor **quark-meson model**, including constituent quarks ψ, three pseudo-scalar pions $\vec{\pi}$ and one scalar field σ (with $\phi \equiv (\sigma, \vec{\pi})$)

$$\Gamma = \int_0^{1/T} \mathrm{d}\tau \int \mathrm{d}^3 z \Big\{ \bar{\psi} \big[\not{\partial} + \gamma^4 \mu + g(\tau_0 \sigma + \mathrm{i}\gamma_5 \vec{\tau} \cdot \vec{\pi}) \big] \psi + + \frac{1}{2}(\partial_\mu \phi)(\partial^\mu \phi) + U(\phi^2/2) - c\sigma \Big\}.$$

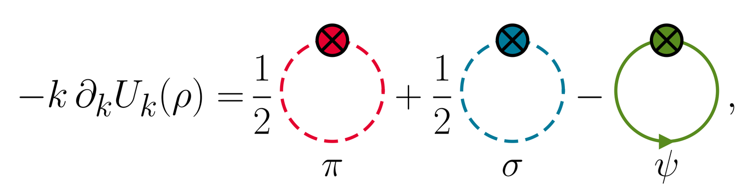
The shape and the minimum of the mesonic self-interaction potential U encodes chiral symmetry breaking, meson masses and the thermodynamics.

The FRG framework allows us to follow the scale evolution of U (and/or other couplings). Here we restrict ourselves to the so called local potential approximation (LPA) where we follow the RG scale (k) evolution of the mesonic self interaction potential U_k only.

Using the exact renormalization group/Wetterich equation

$$\frac{\mathrm{d}\Gamma_k}{\mathrm{d}k} = \frac{1}{2} \mathrm{STr} \Big\{ \big[\Gamma_k^{(2)} + R_k \big]^{-1} \partial_k R_k \Big\}$$

with our model and truncation we obtain the following diagrammatic evolution equation for the potential $U_k(\rho)$



where we evaluated the propagators on a constant homogeneous background

$$\phi \overset{\text{homo.}}{\equiv} (\sqrt{2\rho}, \vec{0}).$$

Numerical methods for flow equations

The RG scale evolution equation for $U_k(\rho)$ constitutes a non-linear, partial integro-differential equation (PDE) in the RG scale k and the field invariant ρ. This PDE can be reformulated and interpreted as a **convection-diffusion equation** [3, 4] by considering the flow of $U_k'(\rho)$,

$$\partial_t u(t,x) + \partial_x \underbrace{F[t, u(t,x)]}_{\pi \,\sim\, \text{convection}} = \partial_x \underbrace{Q[t, x, u(t,x), \partial_x u(t,x)]}_{\sigma \,\sim\, \text{diffusion}} + \partial_x \underbrace{S(t,x)}_{\psi \,\sim\, \text{source}},$$

where we introduced dimensionless variables using the UV initial scale Λ

$$t \equiv -\ln\Big(\frac{k}{\Lambda}\Big), \qquad x \equiv \frac{\rho}{\Lambda^2}, \qquad u(t,x) \equiv \frac{U_k'(\rho)}{\Lambda^2}.$$

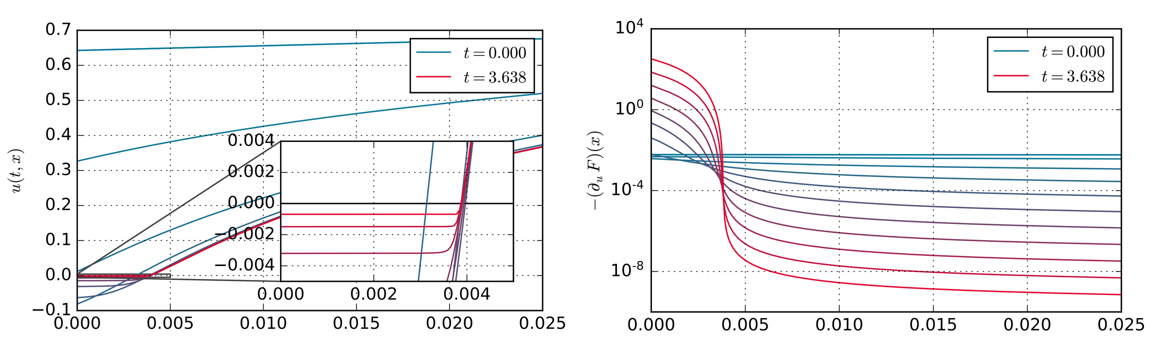


Figure 2: Exemplary vacuum RG scale/time evolution of the derivative of the mesonic potential $u(t,x)$ (left) and corresponding evolution of the approximate *fluid velocity* $(\partial_u F)(x)$ (right). The results shown were obtained using the $O(x^1)$ upwind scheme [5] on 400 equidistant cells in $x \in [0, 0.025]$ for 'spatial' discretization and CVODE's [6] BDF implicit ODE integrators for RG time evolution.

The formulation as a conservation equation strongly suggests the use of established discretization schemes for the numerical solution. For their robustness and relative simplicity we decided to use first- [5] and second-order [7] **finite volume methods** for 'spatial' discretization.

The chiral density wave as an explicit inhomogeneous condensate

The major challenge when working with inhomogeneous condensates in MF and beyond is the computation of propagators – the inversion of the two-point functions – on an inhomogeneous, mesonic background. An inclusion of inhomogeneous chiral condensates renders the fermionic and bosonic two-point functions of the theory non-diagonal in momentum space. Inverting such objects requires significant technical and computational effort.

The so called chiral density wave (CDW) is a very popular ansatz

$$\phi(z) \overset{\text{CDW}}{\equiv} \sqrt{2\rho}\,(\cos(\vec{q}\cdot\vec{z}), 0, 0, \sin(\vec{q}\cdot\vec{z}))$$

for an inhomogeneous condensate in MF computations in the chiral limit ($c = 0$). Using this specific condensate shape and its analytic properties we were able to analytically diagonalize the fermionic and the bosonic two-point functions by means of specifically engineered unitary transformations U_F and U_B, e.g. for the fermionic part

$$U_F(\vec{z}) \equiv \exp\Big(-\frac{\mathrm{i}}{2}\gamma_5 \tau_3 \vec{q}\cdot\vec{z}\Big).$$

Inserting those transformations for the CDW into the exact RG equation yields equations containing only operators, which are diagonal in momentum space, e.g. for the fermionic contribution

$$\frac{\mathrm{d}\Gamma_k^F}{\mathrm{d}k} = -\mathrm{Tr}\Big\{ \big[U_F^\dagger \gamma^4 \Gamma_k^{(1,1,0)} U_F + U_F^\dagger \gamma^4 R_k^F U_F \big]^{-1} U_F^\dagger \partial_k \gamma^4 R_k^F U_F \Big\}.$$

After applying the unitary transformations U_F and U_B we were able to use standard FRG techniques to derive **analytical flow equations for the CDW**.

Renormalization group consistent mean-field results

Using the fermionic part of the LPA flow equation for the CDW we performed a renormalization group consistent MF calculation in the chiral limit using the spatial exponential regulator:

- RG consistent UV completion of the initial condition: $\Gamma_{\Lambda'} \to \Gamma_\Lambda$ [8],
- Consistent parameter fitting: including fermionic vacuum fluctuations in the mesonic two-point functions when fitting the renormalized pion decay constant f_π^{r} and the sigma pole mass m_σ^{p} in vacuum [2].

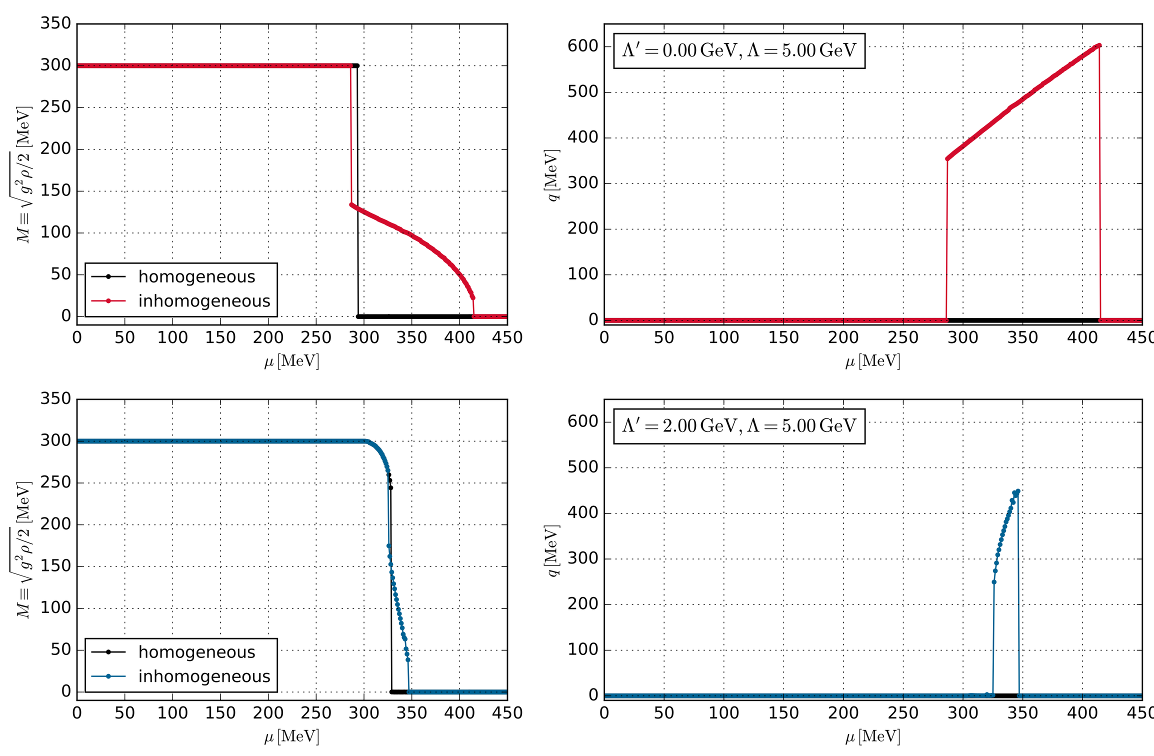


Figure 3: Inhomogeneous window at zero temperature using $f_\pi^{\mathrm{r}} = 88\,\mathrm{MeV}$, $m_\sigma^{\mathrm{p}} = 625\,\mathrm{MeV}$, $m_\pi^{\mathrm{p}} = 0\,\mathrm{MeV}$ and $M = 300\,\mathrm{MeV}$. Upper panel: no-sea/standard MF approximation including no fermionic vacuum fluctuations realized with $\Lambda' = 0\,\mathrm{GeV}$. Lower panel: MF computation including all fermionic vacuum fluctuations below $\Lambda' = 2\,\mathrm{GeV}$. Both RG consistent MF computations use an asymptotically large – compared to μ and T – UV initial scale of $\Lambda = 5\,\mathrm{GeV}$.

Stability analysis of the homogeneous phase

With an advanced Ginzburg-Landau stability analysis of the homogeneous phase, the phase boundary to a phase with inhomogeneous chiral condensation can be detected. This analysis is based on the momentum-dependent mesonic two-point functions extracted from the RG flows. This approach does not rely on specific ansatz functions and was frequently employed in MF studies [1, 9]. In the FRG framework the idea of such a stability analysis was brought forward in [10, 11, 12]. Results shall be presented elsewhere.

Summary

Study of inhomogeneous chiral condensation using the FRG:

- Finite volume methods for FRG flow equations in medium
- Stability analysis of the homogeneous phase
- FRG computation incorporating the CDW condensate

In preparation

- Numerical solution of the full LPA flow eq. for the CDW
- Studying inhomogeneous chiral condensation in lower-dimensional models incorporating various field content
- Calculations beyond the LPA truncation

CRC-TR 211
Strong-interaction matter under extreme conditions

TECHNISCHE UNIVERSITÄT DARMSTADT

GOETHE UNIVERSITÄT FRANKFURT AM MAIN

HGS-HIRe for FAIR
Helmholtz Graduate School for Hadron and Ion Research

Friedrich Naumann STIFTUNG FÜR DIE FREIHEIT

Martin J. Steil
Institut für Kernphysik
Technische Universität Darmstadt
msteil@theorie.ikp.physik.tu-darmstadt.de

Adrian Koenigstein
Institut für theoretische Physik
Goethe-Universität Frankfurt a. M.
koenigstein@th.physik.uni-frankfurt.de

This work is done in close collaboration with J. Braun, M. Buballa, D. H. Rischke and B.-J. Schaefer. This work was supported by the Deutsche Forschungsgemeinschaft (DFG, German Research Foundation) – project number 315477589 – TRR 211. For an overview of all collaborators and details on the project see also *crc-tr211.org* or scan

References

[1] M. Buballa and S. Carignano, "Inhomogeneous chiral condensates", Prog. Part. Nucl. Phys. 81, (2015), 39–96.

[2] S. Carignano, M. Buballa and B.-J. Schaefer, "Inhomogeneous phases in the quark-meson model with vacuum fluctuations", Phys. Rev. D90, (2014) no. 1, 014033.

[3] E. Grossi and N. Wink, "Resolving phase transitions with Discontinuous Galerkin methods", arXiv:1903.09503 [hep-th].

[4] A. Koenigstein, M. J. Steil et al., in preparation.

[5] S. K. Godunov, "A difference method for numerical calculation of discontinuous solutions of the equations of hydrodynamics", Mat. Sb. 47, (1959), 271–306. (in Russian.)

[6] A. C. Hindmarsh et al., "SUNDIALS: Suite of nonlinear and differential/algebraic equation solvers", ACM Trans. Math. Softw. 31, (2005), 363–396.

[7] A. Kurganov and E. Tadmor, "New High-Resolution Central Schemes for Nonlinear Conservation Laws and Convection–Diffusion Equations", J. Comp. Phys. 160, (2000), 241–282.

[8] J. Braun, M. Leonhardt and J. M. Pawlowski, "Renormalization group consistency and low-energy effective theories", SciPost Phys. 6, (2019) no. 5, 056.

[9] S. Rechenberger, A. Koenigstein and M. J. Steil, in preparation.

[10] D. Roscher, J. Braun and J. E. Drut "Phase structure of mass- and spin-imbalanced unitary Fermi gases", Phys. Rev. A91, (2015) no. 5, 053611.

[11] J. Braun et al., "Crystalline ground states in Polyakov-loop extended Nambu–Jona-Lasinio models", Phys. Rev. D93, (2016) no. 1, 014032.

[12] R. A. Tripolt et al. "Low-temperature behavior of the quark-meson model", Phys. Rev. D97, (2018) no. 3, 034022.